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EDITURA

ACADEMIEI OAMENILOR DE ȘTIINȚĂ DIN ROMÂNIA

Annals of the Academy
of Romanian Scientists

Series on Mathematics and its Applications

FOREWORD

Based on a rich scientific tradition, the ***Academy of Romanian Scientists*** *(ARS) is the continuator and the unique heir of the Romanian Academy of Sciences (1936-1948). Then, together with the Academy of Medical Sciences and the Romanian Academy, it was included (by Decree of the Great National Assembly) into the Academy of the Romanian Popular Republic, with Academician Traian Săvulescu as president.*

In 1956, Academician Traian Săvulescu, together with other scientists and members of the Academy, created the Association of the Romanian Scientists, as a partial compensation for the disappearance of the Academy of Romanian Scientists. In 1996, at the first National Congress of the Romanian Scientists (with international participation) the denomination ***Academy of Romanian Scientists*** *was readopted, with the same sigle and the same NGO statute as in 1936.*

By the Decree 52, from January 12, 2007, ARS was recognized as an institution of public interest, situated between the Romanian Academy and the specialized Academies and enjoying the status of chief accountant of public funds.

The Annals of the ***Academy of Romanian Scientists*** *reappeared and continued, during 2006-2007, the tradition from 1936, with one volume every year. Starting with 2008, the Annals are published observing the internationally recognized standards and as several independent series, for each section of ARS.*

It is my real pleasure to congratulate now the members of the Mathematical Section of ARS and the members of the Editorial Board for launching the series on Mathematics and its Applications, of the Annals. To all of them and to the technical staff involved in the production of the journal, my sincere thanks for their work and my best wishes of success in the future activity.

Gen (r). Prof. dr. Vasile Cândea
President of the Academy of Romanian Scientists

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EDITORIAL

The ***Annals of the Academy of Romanian Scientists*** *include scientific journals for all major subject areas of the Academy of Romanian Scientists as a reference source for the scientific community in Romania.*

We are now launching the first number of the series on ***Mathematics and its Applications*** *which joins the already existing series on Information Science and Technology. Other series will be published in the near future to fulfill the mission assumed by ARS.*

We are promoting papers of very good scientific level, making advances in the conceptual understanding and providing new insights into related fields, the basis for future developments. The papers should have a broad appeal to the scientific community and contributions from young scientists are also encouraged. They will be assessed by our referees, trusted researchers in their fields of activity.

On this occasion, I want to thank all members of the ***Editorial Board****, the colleagues who submitted papers or acted as referees and the staff that contributes to the publication of the* ***Annals of ARS****. To all of them, our best wishes of success in this new enterprise and in their activity in general.*

Acad. **Aureliu Sandulescu**

President of the Mathematical Section of ARS

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ANALYSIS AND NUMERICAL APPROACH OF A PIEZOELECTRIC CONTACT PROBLEM*

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Abstract

We consider a mathematical model which describes the frictional contact between an electro-viscoelastic body and a conductive foundation. The contact is modelled with normal compliance and a version of Coulomb's law of dry friction, in which the stiffness and friction coefficients depend on the electric potential. We derive a variational formulation of the problem and, under a smallness assumption, we prove an existence and uniqueness result. The proof is based on arguments on evolutionary variational inequalities and fixed point. Then, we introduce the fully discretized problem and present numerical simulations in the study of a two-dimensional test problem which describe the process of contact in a microelectromechanical swich.

keywords: electro-viscoelastic material, normal compliance, Coulomb's law, variational inequality, weak solution, finite element method, numerical simulations.

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1 Introduction

Contact phenomena involving deformable bodies arise in industry and everyday life and play important roles in structural and mechanical systems. Owning to the complicated surface physics involved, they lead to new and nonstandard mathematical models. Considerable progress has been achieved recently in modelling and mathematical analysis of phenomena of contact and, as a result, a general Mathematical Theory of Contact Mechanics is currently emerging as a discipline on its own right. Its aim is to provide a sound, clear and rigorous background to the construction of models, their variational analysis as well as their numerical simulations, see [9, 16] for details.

Currently there is a considerable interest in contact problems involving piezoelectric materials, i.e. materials characterized by the coupling between the mechanical and electrical properties. This coupling, in a piezoelectric material, leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is applied. The first effect is used in mechanical sensors, and the reverse effect is used in actuators, in engineering control equipments. Piezoelectric materials for which the mechanical properties are elastic are called electro-elastic materials and those for which the mechanical properties are viscoelastic are called electro-viscoelastic materials. General models for electro-elastic materials can be found in [6, 10, 14]. Frictional contact problems for electro-elastic or electro-viscoelastic materials were studied in [7, 12, 13, 17], under the assumption that the foundation is insulated. The results in [7, 12] concern mainly the numerical simulation of the problems while the results in [13, 17] concern the variational formulation of the problems and their unique weak solvability.

The study of mathematical models which describe the evolution of the piezoelectric body in frictional or frictionless contact with a conductive foundation is more recent see, for instance, [3, 4, 5, 11]. The problems studied in [3, 4] are frictionless and describe a dynamic and a quasistatic mechanical process for electro-viscoelastic materials, respectively. The problem studied in [5] is frictional and is modeled with normal compliance and a version of Coulomb's law o dry friction, in which the stiffness and friction coefficients depend on the electric potential; the material is assumed to be electro-elastic and the process is static; an existence and uniqueness result was obtained, a discrete scheme was considered, and numerical simulations were provided.

The problem studied in [11] is frictional, too, and is modeled with the standard normal compliance contact condition and the Coulomb's law of dry friction; the material is assumed to be electro-viscoelastic and the process is quasistatic; an existence and uniqueness result was obtained by using arguments of evolutionary variational inequalitiesd and fixed point.

The results in the present paper are related and parallel our previous results obtained in [5, 11]. Nevertheless, there are several major differences between these papers, that we describe in what follows. First, we recall that in [11] we used the standard normal compliance contact condition and the Coulomb's law of dry friction and, as a result, the mechanical and electrical unknowns are decoupled on the frictional contact conditions. Unlike the problem in [11], in the present paper the electric potential is involved in the frictional contact conditions too, which increase the degree of nonlinearity of the problem and requiers the use of new functionals and operators, different to those used in [11]. Moreover, unlike [11], in the present paper we deal with the numerical approach of the problem and provide numerical simulations. In the present paper we use the boundary conditions on the contact surface used recently in [5] in the study a static process for electro-elastic materials. But, unlike [5], in the present paper we consider a quasistatic process for electro-viscoelastic materials, which leads to an evolutionary model, different from the stationnary model studied in [5].

To conclude, the novelty of this paper consists in the study of a frictional contact problem for electro-viscoelastic materials which takes into account the electric conductivity of the foundation. From the physical point of view, the novelty arises in the fact that we let the frictional contact condition to depend on the electric potential; from the mathematical point of view, the novelty arises in the fact that here we provide the unique solvability of a new model, involving new operators and new functionals, together with its numerical approach and numerical simulations.

The manuscript is structured as follows. In Section 2 we describe the physical setting and present the mathematical model of the contact process. In Section 3 we list the assumption on the problem data, derive the variational formulation of the problem and state our main existence and uniqueness result, Theorem 1. The proof of the theorem is provided in Section 4, based on arguments of evolutionary variational inequalities and fixed point. Finally, in Section 5 we introduce the discretized problem, then we present numerical simulations in the study of a two-dimensional test problem.

2 Problem statement

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal $\boldsymbol{\nu}$. The body is acted upon by body forces of density $\boldsymbol{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_b is prescribed on Γ_b. In the reference configuration the body may come in contact over Γ_3 with an electrically conductive support, the so called foundation. The contact is frictional and we model it with normal compliance and a version of Coulomb's law of dry friction. Also, since the foundation is electrically conductive, we assume that the stiffness coefficient and the friction bound depend on the difference between the electric potential of the body's surface and the electric potential of the foundation. Finally, there may be electrical charges on the part of the body which is in contact with the foundation and which vanish when the contact is lost.

We are interested in the deformation of the body on the time interval $[0, T]$. The process is assumed to be quasistatic, i.e. the inertial effects in the equation of motion are neglected. We denote by $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively and, to simplify the notation, sometimes we do not indicate the dependence of various functions on $\boldsymbol{x}$ or t. In this paper $i, j, k, l = 1, \ldots, d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of $\boldsymbol{x}$, i.e. $f_{,i} = \frac{\partial f}{\partial x_i}$. The dot above a variable represents the time derivatives, i.e. $\dot{f} = \frac{\partial f}{\partial t}$.

We use the notation $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ and "$\cdot$", $\|\cdot\|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$$

for $\boldsymbol{u} = (u_i)$, $\boldsymbol{v} = (v_i) \in \mathbb{R}^d$, and

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$$

for $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

With the notation above, the classical model for the process is as follows.

Problem $\mathcal{P}$. *Find a displacement field $\boldsymbol{u} = (u_i) : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} = (\sigma_{ij}) : \Omega \times [0, T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$ and an electric displacement field $\boldsymbol{D} = (D_i) : \Omega \times [0, T] \to \mathbb{R}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\boldsymbol{E}(\varphi) \qquad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) \qquad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \qquad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1 \times (0, T), \tag{2.5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2 \times (0, T), \tag{2.6}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0, T), \tag{2.7}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = q_b \qquad \text{on } \Gamma_b \times (0, T), \tag{2.8}$$

$$-\sigma_\nu = h_\nu(\varphi - \varphi_0)\, p_\nu(u_\nu - g) \qquad \text{on } \Gamma_3 \times (0, T), \tag{2.9}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau\| \le h_\tau(\varphi - \varphi_0)\, p_\tau(u_\nu - g), \\ &-\boldsymbol{\sigma}_\tau = h_\tau(\varphi - \varphi_0)\, p_\tau(u_\nu - g) \frac{\dot{\boldsymbol{u}}_\tau}{\|\dot{\boldsymbol{u}}_\tau\|} \quad \text{if} \quad \dot{\boldsymbol{u}}_\tau \ne \boldsymbol{0} \end{aligned}\right\} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.10}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = p_e(u_\nu - g)\, h_e(\varphi - \varphi_0) \qquad \text{on } \Gamma_3 \times (0, T), \tag{2.11}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{2.12}$$

We now describe problem (2.1)–(2.12) and provide explanation of the equations and the boundary conditions.

First, equations (2.1) and (2.2) represent the electro-viscoelastic constitutive law in which $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ denotes the linearized strain tensor, $\boldsymbol{E}(\varphi)$ is the electric field, $\mathcal{A}$ and $\mathcal{B}$ are the viscosity and elasticity operators, respectively, $\mathcal{E} = (e_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{E}^*$ is its transpose and $\boldsymbol{\beta}$ denotes the electric permittivity tensor. We recall that $\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2$ and $\boldsymbol{E}(\varphi) = -\nabla\varphi = -(\varphi_{,i})$. Also, the tensors $\mathcal{E}$ and $\mathcal{E}^*$ satisfy the equality

$$\mathcal{E}\boldsymbol{\sigma}\cdot\boldsymbol{v} = \boldsymbol{\sigma}\cdot\mathcal{E}^*\boldsymbol{v} \qquad \forall\boldsymbol{\sigma}\in\mathbb{S}^d,\ \boldsymbol{v}\in\mathbb{R}^d,$$

and the components of the tensor $\mathcal{E}^*$ are given by $e^*_{ijk} = e_{kij}$. Equation (2.1) indicates that the mechanical properties of the materials are described by a viscoelastic Kelvin-Voigt constitutive relation (see [9] for details) which takes into account the dependence of the stress field on the electric field. Relation (2.2) describes a linear dependence of the electric displacement field $\boldsymbol{D}$ on the strain and electric fields; such a relation has been frequently employed in the literature (see, e.g., [6, 7] and the references therein).

Next, equations (2.3) and (2.4) are the balance equations for the stress and electric-displacement fields, respectively, in which "Div" and "div" denote the divergence operators for tensor and vector valued functions, i.e. $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$, $\operatorname{div}\boldsymbol{D} = (D_{i,i})$. We use these equations since the process is assumed to be quasistatic.

Conditions (2.5) and (2.6) are the displacement and traction boundary conditions, whereas (2.7) and (2.8) represent the electric boundary conditions; these conditions show that the displacement field and the electrical potential vanish on Γ_1 and Γ_a, respectively, while the forces and free electric charges are prescribed on Γ_2 and Γ_b, respectively. Also, (2.12) represents the initial condition in which $\boldsymbol{u}_0$ is the given initial displacement field.

We turn to the boundary conditions (2.9)–(2.11), already used in [5], which describe the mechanical and electrical conditions on the potential contact surface Γ_3; there, g represents the gap in the reference configuration between Γ_3 and the foundation, measured along the direction of $\boldsymbol{\nu}$, and φ_0 denotes the electric potential of the foundation.

First, (2.9) represents the normal compliance contact condition in which p_ν is a prescribed nonnegative function which vanishes when its argument is negative and h_ν is a positive function, the stiffness coefficient. Equality (2.9) shows that when there is no contact (i.e. when $u_\nu < g$) then $\sigma_\nu = 0$ and therefore the normal pressure vanishes; when there is contact (i.e. when $u_\nu \geq g$) then $\sigma_\nu \leq 0$ and therefore the reaction of the foundation is towards the body.

Condition (2.10) is the associated friction law where p_τ is a given function and h_τ is the coefficient of friction. According to (2.10) the tangential shear cannot exceed the maximum frictional resistance $h_\tau(\varphi-\varphi_0)\,p_\tau(u_\nu-g)$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and opposes the motion.

Frictional contact conditions of the form (2.9), (2.10) have been used in the study of various piezoelectric contact problems, see, e.g. [11, 17] and the references therein. Unlike these references, we assume here that the stiffness coefficient h_ν and the coefficient of friction h_τ depend on the difference between the potential on the foundation and the body's surface.

Finally, (2.11) is a regularized electrical contact condition on Γ_3, similar to that already used in [3, 4, 5, 11]. Here p_e represents the electrical conductivity coefficient, which vanish when its argument is negative, and h_e is a given function. Thus, condition (2.11) shows that when there is no contact at a point on the surface (i.e. when $u_\nu < g$) then the normal component of the electric displacement field vanishes, and when there is contact (i.e. when $u_\nu \geq g$) then there may be electrical charges which depend on the difference between the potential of the foundation and the body's surface.

Because of the frictional condition (2.10), which is non-smooth, we do not expect the problem to have, in general, any classical solutions. For this reason, we derive in the next section a variational formulation of the problem, then we investigate its weak solvability.

3 Variational formulation

We turn now to the variational formulation of the problem and, to this end, we need additional notation and preliminaries. We use standard notation for the L^p and the Sobolev spaces associated with Ω and Γ; moreover, for a function $\psi \in H^1(\Omega)$ we still write ψ to denote its trace on Γ. Besides the space $L^d(\Omega)^d$ endowed with the canonic inner product $(\cdot,\cdot)_{L^d(\Omega)^d}$ and the associated norm $\|\cdot\|_{L^d(\Omega)^d}$, for the unknowns of Problem $\mathcal{P}$ we use the spaces

$$\begin{aligned}
Q &= \{\ \boldsymbol{\tau} = (\tau_{ij}) :\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \},\\
V &= \{\ \boldsymbol{v} \in H^1(\Omega)^d\ :\ \boldsymbol{v} = \boldsymbol{0}\ \text{ on }\ \Gamma_1\ \},\\
W &= \{\ \psi \in H^1(\Omega) :\ \psi = 0\ \text{ on }\ \Gamma_a\ \}.
\end{aligned}$$

The space Q is a real Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_Q \int_\Omega \sigma_{ij}\tau_{ij}dx$$

and the associated norm $\|\cdot\|_Q$. Also, since $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \qquad (\varphi,\psi)_W = (\nabla\varphi,\nabla\psi)_{L^2(\Omega)^d}$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Moreover, by the Sobolev trace theorem, there exists two positive constants c_0 and $\widetilde{c}_0$ which depend on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\,\|\boldsymbol{v}\|_V \quad \forall\,\boldsymbol{v}\in V, \quad \|\psi\|_{L^2(\Gamma_3)} \le \widetilde{c}_0\,\|\psi\|_W \quad \forall\,\psi\in W. \tag{3.1}$$

Finally, if $(X,\|\cdot\|_X)$ represents a real Banach space, we denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values on X, with the norms

$$\|\boldsymbol{x}\|_{C([0,T];X)} = \max_{t\in[0,T]} \|\boldsymbol{x}(t)\|_X,$$

$$\|\boldsymbol{x}\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|\boldsymbol{x}(t)\|_X + \max_{t\in[0,T]} \|\dot{\boldsymbol{x}}(t)\|_X.$$

Recall that, here and below, the dot represents the derivative with respect to the time variable.

In the study of the mechanical problem (2.1)–(2.12) we assume that the viscosity operator $\mathcal{A}$, the elasticity operator $\mathcal{B}$, the piezoelectric tensor $\mathcal{E}$ and the electric permittivity tensor $\boldsymbol{\beta}$ satisfy

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{A}}>0 \text{ such that}\\
\qquad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_2)\| \le L_{\mathcal{A}}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|\\
\qquad \forall\,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(c) There exists } m_{\mathcal{A}}>0 \text{ such that}\\
\qquad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}_2))\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) \ge m_{\mathcal{A}}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2\\
\qquad \forall\,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega,\\
\qquad \text{for any } \boldsymbol{\xi}\in\mathbb{S}^d.\\
\text{(e) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{3.2}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{B}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{B}}>0 \text{ such that}\\ \qquad \|\mathcal{B}(\boldsymbol{x},\boldsymbol{\xi}_1)-\mathcal{B}(\boldsymbol{x},\boldsymbol{\xi}_2)\|\le L_{\mathcal{B}}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|\\ \qquad \forall\,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbb{S}^d,\ \ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) The mapping } \boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\boldsymbol{\xi}) \text{ is measurable on } \Omega,\\ \qquad \text{for any } \boldsymbol{\xi}\in\mathbb{S}^d.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\mathbf{0}) \text{ belongs to } Q.\end{array}\right. \tag{3.3}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{E}:\Omega\times\mathbb{S}^d\to\mathbb{R}^d.\\ \text{(b) } \mathcal{E}(\boldsymbol{x},\boldsymbol{\tau})=(e_{ijk}(\boldsymbol{x})\tau_{jk})\quad \forall\boldsymbol{\tau}=(\boldsymbol{\tau}_{ij})\in\mathbb{S}^d,\ \ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) } e_{ijk}=e_{ikj}\in L^\infty(\Omega).\end{array}\right. \tag{3.4}$$

$$\left\{\begin{array}{l}\text{(a) } \boldsymbol{\beta}:\Omega\times\mathbb{R}^d\to\mathbb{R}^d.\\ \text{(b) } \boldsymbol{\beta}(\boldsymbol{x},\boldsymbol{E})=(\beta_{ij}(\boldsymbol{x})E_j)\quad \forall\boldsymbol{E}=(E_i)\in\mathbb{R}^d,\ \ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) } \beta_{ij}=\beta_{ji}\in L^\infty(\Omega).\\ \text{(d) There exists } m_\beta>0 \text{ such that } \beta_{ij}(\boldsymbol{x})E_iE_j\ge m_\beta\|\boldsymbol{E}\|^2\\ \qquad \forall\boldsymbol{E}=(E_i)\in\mathbb{R}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\end{array}\right. \tag{3.5}$$

The functions p_r and h_r (for $r=\nu,\ \tau,\ e$) are such that

$$\left\{\begin{array}{l}\text{(a) } p_r:\Gamma_3\times\mathbb{R}\to\mathbb{R}.\\ \text{(b) There exists } L_r>0 \text{ such that}\\ \qquad |p_r(\boldsymbol{x},u_1)-p_r(\boldsymbol{x},u_2)|\le L_r|u_1-u_2|\ \forall u_1,\ u_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(c) There exists } \overline{p}_r\ge 0 \text{ such that}\\ \qquad 0\le p_r(\boldsymbol{x},u)\le\overline{p}_r\ \forall u\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto p_r(\boldsymbol{x},u) \text{ is measurable on } \Gamma_3, \text{ for any } u\in\mathbb{R}.\\ \text{(e) } p_r(\boldsymbol{x},u)=0\quad \forall u<0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{3.6}$$

$$\left\{\begin{array}{l}\text{(a) } h_r:\Gamma_3\times\mathbb{R}\to\mathbb{R},\ \text{for } r=\nu,\ \tau,\ e.\\ \text{(b) There exists } l_r>0 \text{ such that}\\ \qquad |h_r(\boldsymbol{x},\varphi_1)-h_r(\boldsymbol{x},\varphi_2)|\le l_r|\varphi_1-\varphi_2|\\ \qquad \forall\varphi_1,\ \varphi_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3,\ \text{for } r=\nu,\ \tau,\ e.\\ \text{(c) There exists } \overline{h}_r\ge 0 \text{ such that}\\ \qquad 0\le h_r(\boldsymbol{x},\varphi)\le\overline{h}_r\ \forall\varphi\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3,\ \text{for } r=\nu,\ \tau.\\ \text{(d) There exists } \overline{h}_e\ge 0 \text{ such that}\\ \qquad |h_e(\boldsymbol{x},\varphi)|\le\overline{h}_e\ \forall\varphi\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(e) The mapping } \boldsymbol{x}\mapsto h_r(\boldsymbol{x},u) \text{ is measurable on } \Gamma_3,\\ \qquad \text{for any } \varphi\in\mathbb{R},\ \text{for } r=\nu,\ \tau,\ e.\end{array}\right. \tag{3.7}$$

The forces, tractions, volume and surface free charge densities satisfy

$$\boldsymbol{f}_0 \in C([0,T];L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C([0,T];L^2(\Gamma_2)^d), \tag{3.8}$$

$$q_0 \in C([0,T];L^2(\Omega)), \qquad q_b \in C([0,T];L^2(\Gamma_b)). \tag{3.9}$$

Finally, we assume that the gap function, the potential of the foundation and the initial displacement satisfy

$$g \in L^2(\Gamma_3), \quad g \geq 0 \quad \text{a.e. on } \Gamma_3, \tag{3.10}$$

$$\varphi_0 \in L^2(\Gamma_3), \tag{3.11}$$

$$\boldsymbol{u}_0 \in V. \tag{3.12}$$

Next, we define the four mappings $J : W \times V \times V \to \mathbb{R}$, $G : V \times W \times W \to \mathbb{R}$, $\boldsymbol{f} : [0,T] \to V$ and $q : [0,T] \to W$, respectively, by

$$J(\varphi, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} h_\nu(\varphi - \varphi_0)\, p_\nu(u_\nu - g) v_\nu \, da + \int_{\Gamma_3} h_\tau(\varphi - \varphi_0)\, p_\tau(u_\nu - g) \|\boldsymbol{v}_\tau\| \, da, \tag{3.13}$$

$$G(\boldsymbol{u}, \varphi, \psi) = \int_{\Gamma_3} p_e(u_\nu - g)\, h_e(\varphi - \varphi_0) \psi \, da, \tag{3.14}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da, \tag{3.15}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t) \psi \, dx - \int_{\Gamma_b} q_b(t) \psi \, da, \tag{3.16}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$, φ, $\psi \in W$ and $t \in [0,T]$. We note that the definitions of $\boldsymbol{f}$ and q are based on the Riesz representation theorem; moreover, it follows from assumptions (3.6)–(3.11) that the integrals in (3.13)–(3.16) are well-defined and, in addition

$$\boldsymbol{f} \in C([0,T];V), \tag{3.17}$$

$$q \in C([0,T];W). \tag{3.18}$$

Finally, assumptions (3.6) and (3.7) combined with (3.1) yield

$$J(\varphi_1, \boldsymbol{u}_1, \boldsymbol{v}_2) - J(\varphi_1, \boldsymbol{u}_1, \boldsymbol{v}_2) + J(\varphi_2, \boldsymbol{u}_2, \boldsymbol{v}_1) - J(\varphi_2, \boldsymbol{u}_2, \boldsymbol{v}_2) \leq c\,(\|\varphi_1 - \varphi_2\|_W + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V) \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V), \tag{3.19}$$

$$G(\boldsymbol{u}_1, \varphi_1, \psi) - G(\boldsymbol{u}_2, \varphi_2, \psi) \leq (c_0 \tilde{c}_0 L_p \overline{h}_e \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + \tilde{c}_0^2 l_e \overline{p}_e \|\varphi_1 - \varphi_2\|_W) \|\psi\|_W, \tag{3.20}$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in V$, φ_1, φ_2, $\psi \in W$, where $c > 0$.

Using integration by parts, it is straightforward to see that if $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ are sufficiently regular functions which satisfy (2.3)–(2.11) then

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))_Q + J(\varphi(t), \boldsymbol{u}(t), \boldsymbol{v}) - J(\varphi(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \quad (3.21)$$
$$\geq (\boldsymbol{f}(t), \dot{\boldsymbol{u}}(t) - \boldsymbol{v})_V,$$

$$(\boldsymbol{D}(t), \nabla\psi)_{L^2(\Omega)^d} + (q(t), \psi)_W = G(\boldsymbol{u}(t), \varphi(t)), \psi), \quad (3.22)$$

for all $\boldsymbol{v} \in V$, $\psi \in W$ and $t \in [0, T]$. We substitute (2.1) in (3.21), (2.2) in (3.22), note that $\boldsymbol{E}(\varphi) = -\nabla\varphi$ and use the initial condition (2.12). As a result we obtain the following variational formulation of problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. *Find a displacement field* $\boldsymbol{u} : [0, T] \to V$ *and an electric potential* $\varphi : [0, T] \to W$ *such that*

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \varepsilon(\dot{\boldsymbol{u}}(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \quad (3.23)$$
$$+(\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + J(\varphi(t), \boldsymbol{u}(t), \boldsymbol{v}) - J(\varphi(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t))$$
$$\geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V,$$

for all $\boldsymbol{v} \in V$ *and* $t \in [0, T]$,

$$(\boldsymbol{\beta}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\psi)_{L^2(\Omega)^d} \quad (3.24)$$
$$+G(\boldsymbol{u}(t), \varphi(t)), \psi) = (q(t), \psi)_W,$$

for all $\psi \in W$ *and* $t \in [0, T]$, *and*

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \quad (3.25)$$

To study problem $\mathcal{P}_V$ we make the smallness assumption

$$\tilde{c}_0^2 l_e \overline{p}_e < m_\beta, \quad (3.26)$$

where $\tilde{c}_0$, l_e, $\overline{p}_e$ and m_β are given in (3.1) (3.7), (3.6) and (3.5), respectively. We note that only the trace constant, the Lipshitz contant of h_e, the bound of p_e and the coercivity constant of $\boldsymbol{\beta}$ and are involved in (3.26); therefore, this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of the problem. Moreover, it is satisfied when the obstacle is insulated, since then $p_e \equiv 0$ and so $\overline{p}_e = 0$.

Our main existence and uniqueness result that we state now and prove in the next section is the following.

Theorem 1. *Assume that* (3.2)–(3.12) *and* (3.26) *hold. Then Problem* $\mathcal{P}_V$ *has a unique solution which satisfies*

$$\boldsymbol{u} \in C^1([0,T];V), \quad \varphi \in C([0,T];W). \tag{3.27}$$

A quadruple of functions $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ which satisfies (2.1), (2.2), (3.23)–(3.25) is called a *weak solution* of the piezoelectric contact problem $\mathcal{P}$. It follows from Theorem 1 that, under the assumptions (3.2)–(3.12), (3.26), there exists a unique weak solution of Problem $\mathcal{P}$. To precise the regularity of the weak solution we note that the constitutive relations (2.1) and (2.2), assumptions (3.2)–(3.5) and regularity (3.27) imply that

$$\boldsymbol{\sigma} \in C([0,T];Q), \qquad \boldsymbol{D} \in C([0,T];L^2(\Omega)^d). \tag{3.28}$$

Moreover, using again (2.1), (2.2) combined with (3.23), (3.24) and the notation (3.13)–(3.16), after standard arguments we obtain that $\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0}$ and $\mathrm{div}\,\boldsymbol{D}(t) = q_0(t)$, for all $t \in [0,T]$. It follows now from the regularity (3.8) and (3.9) that

$$\mathrm{Div}\,\boldsymbol{\sigma} \in C([0,T];L^2(\Omega)^d), \qquad \mathrm{div}\,\boldsymbol{D} \in C([0,T];L^2(\Omega)). \tag{3.29}$$

We conclude that the weak solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ of the piezoelectric contact problem $\mathcal{P}$ has the regularity (3.27)–(3.29).

4 Proof of Theorem 1

We turn now to the proof of Theorem 1 which will be carried out in several steps. We assume in what follows that (3.2)–(3.12) and (3.26) hold and, everywhere below, we denote by c various positive constants which are independent on time and whose value may change from line to line. We consider the product space $X = V \times W$ together with the inner product

$$(\boldsymbol{x}, \boldsymbol{y})_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_W \qquad \forall \boldsymbol{x} = (\boldsymbol{u}, \varphi),\ \boldsymbol{y} = (\boldsymbol{v}, \psi) \in X$$

and the associated norm $\|\cdot\|_X$. Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \eta_2) \in C([0,T],X)$ be given. In the first step, we consider the following intermediate problem.

Problem $\mathcal{P}_\eta^{disp}$. *Find a displacement field* $\boldsymbol{u}_\eta : [0,T] \to V$ *such that*

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q \\
&\quad + (\mathcal{E}^*\nabla\eta_2(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q + J(\eta_2(t), \boldsymbol{\eta}_1(t), \boldsymbol{v}) \\
&\qquad - J(\eta_2(t), \boldsymbol{\eta}_1(t), \dot{\boldsymbol{u}}_\eta(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t))_V \quad \forall \boldsymbol{v} \in V,\ t \in [0,T],
\end{aligned} \tag{4.1}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.2}$$

In the study of the variational problem $\mathcal{P}_\eta^{disp}$ we have the following result.

Lemma 1. *There exists a unique solution* $\boldsymbol{u}_\eta \in C^1([0,T],V)$ *to the problem* (4.1)–(4.2)*. Moreover, if* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$ *are two solutions of problem* (4.1)–(4.2) *corresponding to the data* $\boldsymbol{\eta}^1 = (\boldsymbol{\eta}_1^1, \eta_2^1)$*,* $\boldsymbol{\eta}^2 = (\boldsymbol{\eta}_1^2, \eta_2^2) \in C([0,T],X)$ *then there exists* $c > 0$ *such that*

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V \le c\,\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_X \quad \forall\, t \in [0,T]. \tag{4.3}$$

Proof. We use classical results on elliptic variational inequalities (see [9, p. 60]) to deduce that, for each $t \in [0,T]$, there exists a unique element $\boldsymbol{v}_\eta(t) \in V$ such that

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)))_Q \\ +(\mathcal{E}^*\nabla\eta_2(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)))_Q + J(\eta_2(t), \boldsymbol{\eta}_1(t), \boldsymbol{v}) \\ -J(\eta_2(t), \boldsymbol{\eta}_1(t), \boldsymbol{v}_\eta(t)) \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}_\eta(t))_V \quad \forall\, \boldsymbol{v} \in V.\end{aligned} \tag{4.4}$$

Let $t_1,\ t_2 \in [0,T]$; using (4.4) for $t = t_1$ and $t = t_2$, we easily derive the inequality

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_1)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_1)) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_2)))_Q \\ &\le (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(t_1)) - \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_1(t_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_2)) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_1)))_Q + \\ &\quad +(\mathcal{E}^*\nabla\eta_2(t_1) - \mathcal{E}^*\nabla\eta_2(t_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_2)) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t_1)))_Q \\ &\quad +J(\eta_2(t_1), \boldsymbol{\eta}_1(t_1), \boldsymbol{v}_\eta(t_2)) - J(\eta_2(t_1), \boldsymbol{\eta}_1(t_1), \boldsymbol{v}_\eta(t_1)) \\ &\quad +J(\eta_2(t_2), \boldsymbol{\eta}_1(t_2), \boldsymbol{v}_\eta(t_1)) - J(\eta_2(t_2), \boldsymbol{\eta}_1(t_2), \boldsymbol{v}_\eta(t_2)) \\ &\quad +(\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2), \boldsymbol{v}_\eta(t_1) - \boldsymbol{v}_\eta(t_2))_V.\end{aligned}$$

Then, we use assumptions (3.2), (3.3), (3.4) and (3.19) to obtain

$$\begin{aligned}\|\boldsymbol{v}_\eta(t_1) - \boldsymbol{v}_\eta(t_2)\|_V \le c\,(\|\boldsymbol{\eta}_1(t_1) - \boldsymbol{\eta}_1(t_2)\|_V \\ +\|\eta_2(t_1) - \eta_2(t_2)\|_W + \|\boldsymbol{f}(t_1) - \boldsymbol{f}(t_2)\|_V).\end{aligned} \tag{4.5}$$

From (4.5), (3.17) and the regularity of $\boldsymbol{\eta}$ it follows that $\boldsymbol{v}_\eta \in C([0,T];V)$. Let $\boldsymbol{u}_\eta : [0,T] \to V$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)\,ds + \boldsymbol{u}_0, \qquad \forall t \in [0,T]. \tag{4.6}$$

It follows from (4.6) and (4.4) that $\boldsymbol{u}_\eta$ is a solution of Problem $\mathcal{P}_\eta^{disp}$ and, moreover, $\boldsymbol{u}_\eta \in C^1([0,T];V)$. This proves the existence part of Lemma 1. The uniqueness part follows from the unique solvability of the variational inequality (4.4) at each $t \in [0,T]$.

Let now denote by $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ the solutions of problem (4.1)–(4.2) corresponding to the data $\boldsymbol{\eta}^1 = (\boldsymbol{\eta}_1^1, \eta_2^1)$, $\boldsymbol{\eta}^2 = (\boldsymbol{\eta}_1^2, \eta_2^2) \in C([0,T],X)$ and let $\dot{\boldsymbol{u}}_1 = \boldsymbol{v}_1$, $\dot{\boldsymbol{u}}_2 = \boldsymbol{v}_2$. Arguments similar to those used in the proof of (4.5) lead to

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V \le c\,(\|\boldsymbol{\eta}_1^1(t) - \boldsymbol{\eta}_1^2(t)\|_V + \|\eta_2^1(t) - \eta_2^2(t)\|_V) \quad \forall\, t \in [0,T],$$

which shows that (4.3) holds. □

In the next step we use the solution $\boldsymbol{u}_\eta \in C^1([0,T],V)$ obtained in Lemma 1 to construct the following variational problem.

Problem $\mathcal{P}_\eta^{pot}$. *Find an electric potential field $\varphi_\eta : [0,T] \to W$ such that*

$$\begin{aligned}(\boldsymbol{\beta}\nabla\varphi_\eta(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \nabla\psi)_{L^2(\Omega)^d}\\ +G(\boldsymbol{u}_\eta(t), \eta_2(t), \psi) = (q(t), \psi)_W \qquad \forall\, \psi \in W,\ t \in [0,T]\end{aligned} \tag{4.7}$$

The well-posedness of the problem $\mathcal{P}_\eta^{pot}$ is given by the following result.

Lemma 2. *There exists a unique solution $\varphi_\eta \in C([0,T];W)$ which satisfies* (4.7)*. Moreover, if $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and φ_1, φ_2 are two solutions of of* (4.1)–(4.2) *and* (4.7)*, respectively, corresponding to $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0,T];X)$, then there exists $c > 0$ such that*

$$\begin{aligned}\|\varphi_1(t) - \varphi_2(t)\|_W \le c\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V\\ +\frac{\tilde{c}_0^2 l_e \overline{p}_e}{m_\beta}\,\|\boldsymbol{\eta}^1(t) - \boldsymbol{\eta}^2(t)\|_X \quad \forall\, t \in [0,T].\end{aligned} \tag{4.8}$$

Proof. It follows from (3.5) that the bilinear form

$$a(\varphi, \psi) = (\boldsymbol{\beta}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} \tag{4.9}$$

is continuous, symmetric and coercive on W. Moreover, using (3.18), (3.20), assumption (3.4) on the piezoelectric tensor $\mathcal{E}$ and the regularity $\boldsymbol{u}_\eta \in C^1([0,T];V)$, it follows that the function $q_\eta : [0,T] \to W$, given by

$$\begin{aligned}(q_\eta(t), \psi)_W = (q(t), \psi)_W + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \nabla\psi)_{L^2(\Omega)^d}\\ -G(\boldsymbol{u}_\eta(t), \eta_2(t), \psi) \qquad \forall\, \psi \in W,\ t \in [0,T],\end{aligned} \tag{4.10}$$

is continuous. The existence and uniqueness part in Lemma 4.3 is now a straight consequence of the well-known Lax-Milgram theorem applied to the time-dependent variational equation

$$a(\varphi(t),\psi) = (q_\eta(t),\psi) \qquad \forall \psi \in W,\ t \in [0,T],$$

combined with the equalities (4.9), (4.10). Moreover, the estimate (4.8) follows from (4.7), (3.4), (3.5) and (3.20). □

We now consider the operator $\Lambda : C([0,T];X) \to C([0,T];X)$ defined by

$$\Lambda\boldsymbol{\eta}(t) = (\boldsymbol{u}_\eta(t),\varphi_\eta(t)) \qquad \forall\,\boldsymbol{\eta} \in C([0,T];X)),\ t \in [0,T]. \tag{4.11}$$

The next step consists in the following result.

Lemma 3. *There exists a unique $\boldsymbol{\eta}^* \in C([0,T];X)$ such that $\Lambda\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$.*

Proof. Let $\boldsymbol{\eta}^1 = (\boldsymbol{\eta}_1^1,\eta_2^1)$, $\boldsymbol{\eta}^2 = (\boldsymbol{\eta}_1^2,\eta_2^2) \in C([0,T];X)$ and, for simplicity, we use the notation $\boldsymbol{u}_i$ and φ_i for the functions $\boldsymbol{u}_{\eta_i}$ and φ_{η_i} obtained in Lemmas 1 and 2, for $i = 1,2$. Let $t \in [0,T]$. Using (4.11) and (4.8) we obtain

$$\|\Lambda\boldsymbol{\eta}^1(t) - \Lambda\boldsymbol{\eta}^2(t)\|_Q \le c\,\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \frac{\tilde{c}_0^2 l_e \overline{p}_e}{m_\beta}\,\|\boldsymbol{\eta}^1(t) - \boldsymbol{\eta}^2(t)\|_X. \tag{4.12}$$

On the other hand, since

$$\boldsymbol{u}_i(t) = \boldsymbol{u}_0 + \int_0^t \dot{\boldsymbol{u}}_i(s)\,ds$$

we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \le \int_0^t \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|_V\,ds$$

and, combining this inequality with (4.3), we find

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \le c\int_0^t \|\boldsymbol{\eta}^1(t) - \boldsymbol{\eta}^2(t)\|_X\,ds. \tag{4.13}$$

We use now (4.12) and (4.13) to obtain

$$\|\Lambda\boldsymbol{\eta}^1(t) - \Lambda\boldsymbol{\eta}^2(t)\|_Q \le c\int_0^t \|\boldsymbol{\eta}^1(s) - \boldsymbol{\eta}^2(s)\|_X\,ds + \frac{\tilde{c}_0^2 l_e \overline{p}_e}{m_\beta}\,\|\boldsymbol{\eta}^1(t) - \boldsymbol{\eta}^2(t)\|_X.$$

The last inequality combined with the smallness assumption (3.26) allows the use of Corollary 2.1 in [18]; as a result it follows that the operator Λ has a unique fixed point, which concludes the proof. $\square$

We have now all the ingredients to prove the Theorem 1.

Existence. Let $\boldsymbol{\eta}^* = (\boldsymbol{\eta}_1^*, \eta_2^*) \in C([0,T]; X)$ be the fixed point of the operator Λ, and let $\boldsymbol{u}_{\eta^*}$, φ_{η^*} be the solutions of problems $\mathcal{P}_\eta^{disp}$ and $\mathcal{P}_\eta^{pot}$, respectively, for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. It follows from (4.11) that $\boldsymbol{u}_{\eta^*} = \boldsymbol{\eta}_1^*$, $\varphi_{\eta^*} = \eta_2^*$ and therefore (4.1), (4.2) and (4.7) imply that $(\boldsymbol{u}_{\eta^*}, \varphi_{\eta^*})$ is a solution of problem $\mathcal{P}_V$. The regularity (3.27) follows from Lemmas 4.2 and 4.3.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ, given by Lemma 3. $\square$

5 Numerical approach

Discretized problem. Everywhere below we assume that (3.2)–(3.12) and (3.26) hold. We now introduce a fully discrete scheme to approximate the solution of Problem $\mathcal{P}_V$, provided by Theorem 1. First, we consider two finite dimensional spaces $V^h \subset V$ and $W^h \subset W$ approximating the spaces V and W, respectively, in which $h > 0$ denotes the spatial discretization parameter. In the numerical simulations presented below, V^h and W^h consist of continuous and piecewise affine functions, that is,

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \,:\, \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d \;\forall\, Tr \in \mathcal{T}^h,\ \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma_1\}, \tag{5.1}$$

$$W^h = \{\zeta^h \in C(\overline{\Omega}) \,:\, \zeta^h_{|Tr} \in P_1(Tr) \;\;\forall\, Tr \in \mathcal{T}^h,\ \zeta^h = 0 \text{ on } \Gamma_a\}, \tag{5.2}$$

where Ω is assumed to be a polygonal domain, $\mathcal{T}^h$ denotes a finite element triangulation of $\overline{\Omega}$, and $P_1(Tr)$ represents the space of polynomials of global degree less or equal to one in Tr. In addition, we consider a uniform partition of $[0,T]$, $0 = t_0 < t_1 < \ldots < t_N = T$, that we use to discretize the time derivatives and, everywhere in this section, we use the notation k for the time step size, i.e. $k = T/N$. Finally, for a continuous function $f(t)$ we denote $f_n = f(t_n)$ and for a sequence $\{w_n\}_{n=0}^N$ we use $\delta w_n = (w_n - w_{n-1})/k$ for the divided differences.

Let $\boldsymbol{u}_0^{hk}$ be an appropriate approximation of the initial condition $\boldsymbol{u}_0$. Then using the backward Euler scheme, the fully discrete approximation of Problem $\mathcal{P}_V$ is the following.

Problem $\mathcal{P}_V^{hk}$. *Find a discrete displacement field* $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset K^h$ *and a discrete electric potential* $\varphi^{hk} = \{\varphi_n^{hk}\}_{n=0}^N \subset W^h$ *such that*

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q\\ &\quad +(\mathcal{E}^*\nabla\varphi_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{w}^h) - \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))_Q + J(\varphi_n^{hk}, \boldsymbol{u}_n^{hk}, \boldsymbol{w}^h) - J(\varphi_n^{hk}, \boldsymbol{u}_n^{hk}, \delta\boldsymbol{u}_n^{hk})\\ &\quad\quad \geq (\boldsymbol{f}_n, \boldsymbol{w}^h - \boldsymbol{u}_n^{hk})_V \quad \forall\, \boldsymbol{w}^h \in V^h, \text{ for all } n = 1, \dots, N,\end{aligned}$$

$$\begin{aligned}&(\boldsymbol{\beta}\nabla\varphi_n^{hk}, \nabla\psi^h)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \nabla\psi^h)_{L^2(\Omega)^d} + G(\boldsymbol{u}_n^{hk}, \varphi_n^{hk}, \psi^h)\\ &\quad\quad = (q_n, \psi^h)_W \quad \forall\, \psi^h \in W^h, \text{ for all } n = 0, \dots, N.\end{aligned}$$

The existence of a unique solution to Problem $\mathcal{P}_V^{hk}$ can be obtained by arguments similar to those presented in Section 4. The solution algorithm in solving Problem $\mathcal{P}_V^{hk}$ combines the finite differences method (the backward Euler difference method) with the linear iterations method (the Newton method). Details on these methods can be find in the monograph [19] and, therefore, we omit them. Nevertheless, we note that the numerical treatment of the frictional contact term is based on the use of a penalization method for the contact part and an augmented Lagrangean method for the non-smooth friction part, see [19] and [2], respectively.

Numerical simulations. We now present numerical simulations in the study of a real-world example of Problem $\mathcal{P}$, the microelectromechanical switches, see [15] for details. Microelectromechanical systems (MEMS) are being recognized as enabling components to switch or tune radio frequency (rf) components, modules or systems in manufacturing and operation. In short, they are referred to as rf-MEMS. Most rf-MEMS involve the manipulation of air as the dielectric materials. Various designs of capacitive rf-MEMS switches made out of nickel, aluminium, gold or zinc oxide have so far been reported in literature, see for instance [1, 8]. The mechanical simulation of switch consists in the following design concept: the switch design is based on a suspended metal bridge (zinc oxyde in our example) which connects two grounds of a coplanar wave-guide and crosses over a signal line on which a dielectric foundation is deposited. When an external force is acting, this action pulls the metal bridge down and contacts the dielectric, which results in a low impedance between signal line and ground line for shunting high-frequency signal transmission.

To describe this example, we consider an electro-viscoelastic body extended indefinitely in the direction X_1 of a cartesian coordinate system

(O, X_1, X_2, X_3). The material used is assumed to be a linearly isotropic piezoceramic with hexagonal symmetry like zinc oxyde material (class $6mm$ in the international classification [10]) which presents a viscous behavior. In the crystallographic frame, the X_3-direction is a six-fold revolution symmetry axis and the (X_1OX_3) and (X_2OX_3) planes are mirrors. The electrical and mechanical loads applied to the body are supposed to be constant along the X_1 direction. As a consequence, the fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ turn out to be constant along X_1. In addition, we suppose that $\boldsymbol{\varepsilon}_{11} = 0$, $\boldsymbol{\varepsilon}_{12} = 0$, $\boldsymbol{\varepsilon}_{13} = 0$ and $D_1 = 0$, i.e. we consider a plane problem. Under these assumptions, the unknown of our electro-viscoelastic contact problem is the pair $(\boldsymbol{u}, \varphi)$ where the displacement field $\boldsymbol{u} = (u_2, u_3)$ belongs to the plane (O, X_2, X_3).

Assume that the viscosity and elasticity operators are linear and denote by a_{ijkl} and b_{ijkl} their components, i.e. $\mathcal{A} = (a_{ijkl})$ and $\mathcal{B} = (b_{ijkl})$. Then, in the system (O, X_2, X_3), the constitutive equations (2.1) and (2.2) can be written by using the following compressed matrix notation,

$$\begin{bmatrix} \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} b_{22} & b_{23} & 0 & 0 & e_{32} \\ b_{23} & b_{33} & 0 & 0 & e_{33} \\ 0 & 0 & b_{44} & e_{24} & 0 \\ 0 & 0 & e_{24} & -\beta_{22} & 0 \\ e_{32} & e_{33} & 0 & 0 & -\beta_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ -E_2 \\ -E_3 \end{bmatrix} + \begin{bmatrix} a_{22} & a_{23} & 0 & 0 & 0 \\ a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\varepsilon}_{22} \\ \dot{\varepsilon}_{33} \\ 2\dot{\varepsilon}_{23} \\ -E_2 \\ -E_3 \end{bmatrix}. \tag{5.3}$$

Here $\dot{\varepsilon}_{ij} = \dfrac{1}{2}\left(\dfrac{\partial \dot{u}_i}{\partial x_j} + \dfrac{\partial \dot{u}_j}{\partial x_i}\right)$ and note that equation (5.3) is obtained by the identification

$$b_{ijkl} \equiv b_{pq} = \begin{pmatrix} b_{22} & b_{23} & 0 \\ b_{23} & b_{33} & 0 \\ 0 & 0 & b_{44} \end{pmatrix}, \quad a_{ijkl} \equiv a_{pq} = \begin{pmatrix} a_{22} & a_{23} & 0 \\ a_{23} & a_{33} & 0 \\ 0 & 0 & a_{44}, \end{pmatrix},$$

with the rule

$$\begin{aligned} ij \text{ or } kl = 22 \quad &\longrightarrow \quad p \text{ or } q = 2, \\ ij \text{ or } kl = 33 \quad &\longrightarrow \quad p \text{ or } q = 3, \\ ij \text{ or } kl = 23 \text{ or } 32 \quad &\longrightarrow \quad p \text{ or } q = 4. \end{aligned}$$

This rule, which allows to describe the link between the fourth-order tensors of components b_{ijkl} and a_{ijkl} and the corresponding second-order tensors of components b_{pq} and a_{pq}, respectively, is obtained by using the symmetries of the various tensors involved in the constitutive law. In the same way, for the third order piezoelectric tensor we have

$$e_{ijk} \equiv e_{iq} = \begin{pmatrix} 0 & 0 & e_{24} \\ e_{32} & e_{33} & 0 \end{pmatrix} \quad \text{with} \quad \begin{array}{rcl} jk = 22 & \longrightarrow & q = 2, \\ jk = 33 & \longrightarrow & q = 3, \\ jk = 23 \text{ or } 32 & \longrightarrow & q = 4. \end{array}$$

We use the material constants given in Tables 1 and 2, in which $\epsilon_0 8.885 \times 10^{-12} C^2/Nm^2$ represents the permittivity constant of the vacuum.

Elastic (GPa)				Viscoelastic $(GPa \cdot s)$			
b_{22}	b_{23}	b_{33}	b_{44}	a_{22}	a_{23}	a_{33}	a_{44}
210	105	211	42.5	2.1	1.05	2.11	0.425

Table 1: Elastic and viscoelastic constants of the piezoelectric body.

Piezoelectric (C/m^2)			Permittivity (C^2/Nm^2)	
e_{32}	e_{33}	e_{24}	β_{22}/ϵ_0	β_{33}/ϵ_0
-0.61	1.14	-0.59	-8.3	-8.8

Table 2: Electric constants of the piezoelectric body.

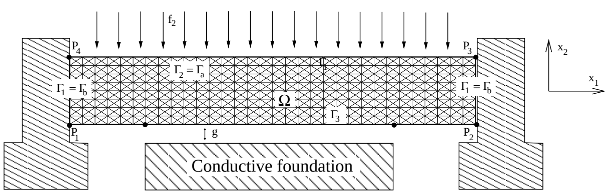

Figure 1: Physical setting of MEMS : an electroelastic body in contact with a conductive obstacle.

As a two-dimensional example, we consider the physical setting depicted in Figure 1, where $\Omega = [0,12] \times [0,2]$, $\Gamma_1 = \Gamma_b = (\{0\} \times [0,2]) \cup (\{12\} \times [0,2])$, $\Gamma_2 = \Gamma_a = ([0,12] \times \{2\}) \cup ([0,2] \times \{0\}) \cup ([10,12] \times \{0\})$, and the potential contact surface is $\Gamma_3 = [2,10] \times \{0\}$. The body is subjected to the action of surface pression $\boldsymbol{f}_2 = (0, -5)N/\mu m$ which acts on the top of the bridge, i.e. on $[0, 12] \times \{2\}$; the body forces and electric charges vanish, i.e. $\boldsymbol{f}_0 = \mathbf{0}\, N/\mu m^2$, $q_0 = 0\, C/\mu m^2$ and $q_b = 0\, C/\mu m$; and the gap between the body and the foundation is $g = 0.5\mu m$. The functions h_r and p_r $(r = \nu, \tau)$ in the frictional contact conditions (2.9) and (2.10) are given by

$$h_r(s) = c_r \times \begin{cases} \alpha_r & \text{if } |s| > 128, \\ 1 + (\alpha_r - 1) \times \frac{|s|}{128} & \text{if } |s| \leq 128, \end{cases}$$

$$p_r(s) = \begin{cases} 0 & \text{if } s < 0, \\ s & \text{if } 0 \leq s \leq n_\nu, \\ n_r & \text{if } s > n_\nu, \end{cases}$$

where c_r, α_r and n_r are positive constants, $\alpha_r > 1$. And, finally, for the regularized electrical condition (2.11) we take

$$h_e(s) = \begin{cases} -m_e & \text{if } s < -m_e, \\ s & \text{if } -m_e \leq s \leq m_e, \\ m_e & \text{if } s > m_e \end{cases} \qquad p_e(s) = k_e \times \begin{cases} 0 & \text{if } s < 0, \\ \frac{s}{\epsilon_e} & \text{if } 0 \leq s \leq \epsilon_e, \\ 1 & \text{if } s > \epsilon_e, \end{cases}$$

where m_e, k_e and ϵ_e are positive constants.

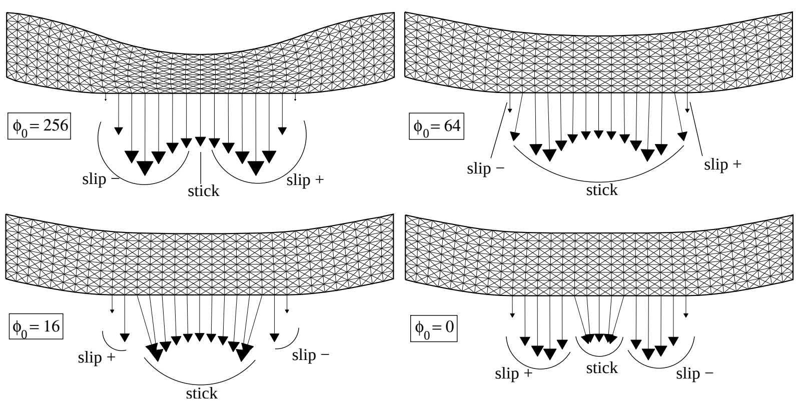

Figure 2: Sequence of deformed meshes and corresponding contact forces.

Our interest in this piezoelectric contact model is to study the influence of the electric potential of the foundation on the process. Our results are presented in Figures 2–6, in which we use the notation $\phi_0 = -\varphi_0$ and $k = k_e$.

In Figure 2 we plot a sequence of deformed meshes with the corresponding contact interface forces and the contact status, obtained for four different values of the electric potential of the foundation: $\varphi_0 = -\phi_0$, where ϕ_0 takes successively the values 256, 64, 16 and 0. It results from the figure that the deformations and the magnitude of the contact forces decrease when ϕ_0 decreases, i.e. when the magnitude of the electric potential of the foundation decreases.

According to Figure 3 we note that, for k given, the magnitude of the normal electric displacement increases with ϕ_0. A similar behavior follows from Figure 4 which shows that, for a given ϕ_0, the magnitude of the normal electric displacement increases with the electrical conductivity coefficient k. These results are compatible with the electrical boundary condition we use on the contact surface and show the effect of the conductivity of the foundation on the process.

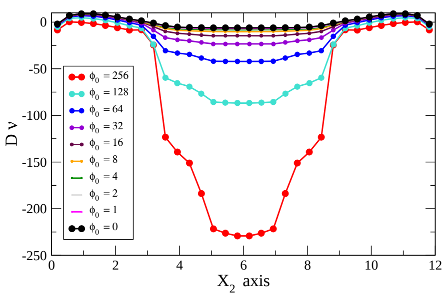

Figure 3: Dependence of the normal electric displacement $\boldsymbol{D}\cdot\boldsymbol{\nu}$ with respect to ϕ_0, for $k = 1$.

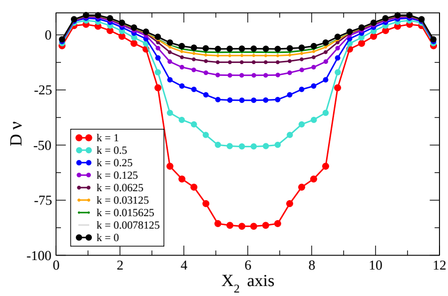

Figure 4: Dependence of the normal electric displacement $\boldsymbol{D}\cdot\boldsymbol{\nu}$ with respect to k, for $\phi_0 = 128$.

Finally, Figure 5 shows the electric potential in the body whereas Figure 6 represents the electric displacement fields in the deformed configuration, for four different values of the potential of the foundation, corresponding to $\phi_0 = 256$, $\phi_0 = 64$, $\phi_0 = 16$ and $\phi_0 = 0$. According to Figures 5 and 6, we note that the magnitude of the electric potential and the magnitude of the electric displacement increase on the contact interface, when the magnitude ϕ_0 of the potential of the foundation increases.

We conclude that our simulations above underline the effects of the electrical conductivity of the foundation on the frictional contact process. Performing these simulations we found that the numerical solution worked well

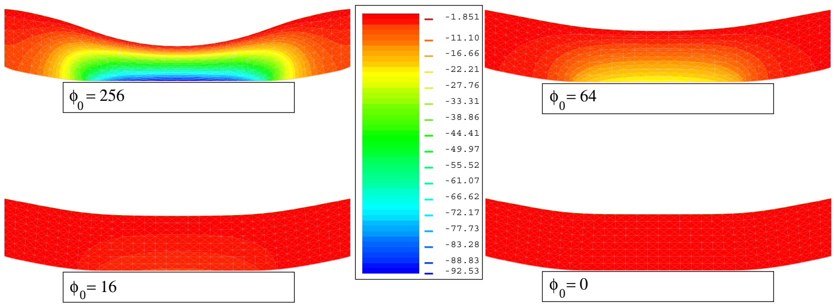

Figure 5: Sequence of deformed meshes and corresponding electric potential.

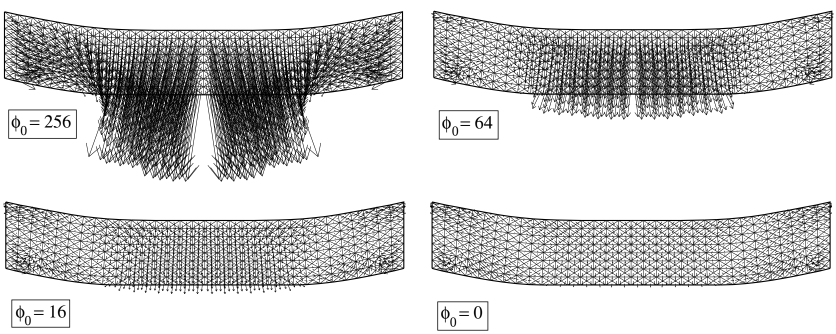

Figure 6: Sequence of deformed meshes and corresponding electric displacement fields.

and the convergence was rapid.

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SINGULARLY PERTURBED CAUCHY PROBLEM FOR ABSTRACT LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN HILBERT SPACES*

Andrei Perjan† Galina Rusu‡

Abstract

We study the behavior of solutions to the problem

$$\begin{cases} \varepsilon\left(u''_\varepsilon(t) + A_1 u_\varepsilon(t)\right) + u'_\varepsilon(t) + A_0 u_\varepsilon(t) = f_\varepsilon(t), \quad t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u'_\varepsilon(0) = u_{1\varepsilon}, \end{cases}$$

as $\varepsilon \to 0$, where A_1 and A_0 are two linear self-adjoint operators in a Hilbert space H.

MSC: 35B25, 35K15, 35L15, 34G10

keywords: singular perturbations; Cauchy problem; boundary layer function.

1 Introduction

Let H be a real Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$. Let $A_i : D(A_i) \to H$, $i = 0, 1$, be two linear self-adjoint operators.

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Consider the following Cauchy problem:

$$\begin{cases} \varepsilon\left(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\right) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f_\varepsilon(t), & t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter($\varepsilon \ll 1$), $u_\varepsilon, f_\varepsilon : [0,T) \to H$.

We will investigate the behavior of solutions $u_\varepsilon(t)$ to the perturbed system (P_ε) when $\varepsilon \to 0$, $u_{0\varepsilon} \to u_0$ and $f_\varepsilon \to f$. We will establish a relationship between solutions to the problem (P_ε) and the corresponding solutions to the following unperturbed system:

$$\begin{cases} v'(t) + A_0 v(t) = f(t), & t \in (0,T), \\ v(0) = u_0. \end{cases} \tag{P_0}$$

In our study we will use the following conditions:

(H1) *The operator $A_0 : D(A_0) \subseteq H \to H$ is self-adjoint and positive defined, i.e. there exists $\omega_0 > 0$ such that*

$$(A_0 u, u) \geq \omega_0 \, |u|^2, \quad \forall u \in D(A_0);$$

(H2) *The operator $A_1 : D(A_1) \subseteq H \to H$ is self-adjoint, $D(A_0) \subseteq D(A_1)$ and there exists $\omega_1 > 0$ such that*

$$|(A_1 u, u)| \leq \omega_1 \, (A_0 u, u)\,, \quad \forall u \in D(A_0).$$

If, in some topology, $u_\varepsilon(t)$ tends to the corresponding solutions $v(t)$ of the unperturbed system (P_0) as $\varepsilon \to 0$, then the system (P_0) is called *regularly perturbed*. In the opposite case system (P_0) is called *singularly perturbed*. In the last case, a subset of $[0, \infty)$, in which the solution $u_\varepsilon(t)$ has a singular behavior relative to ε, arises. This subset is called *the boundary layer*. The function which defines the singular behavior of the solution $u_\varepsilon(t)$ within the boundary layer is called *the boundary layer function*.

Many physical processes are described by systems of type (P_ε). For example, the equation

$$\rho v_{tt} + \gamma v_t = \sigma \Delta v$$

(where ρ, γ, σ are the mass density per unit area of the membrane, the coefficient of viscosity of the medium, and the tension of the membrane, respectively), which characterizes the vibration of a membrane in a viscous medium, can be rewritten as

$$\varepsilon^2 u_{tt} + u_t = \Delta u,$$

with $\varepsilon = (\rho\sigma)^{1/2}/\gamma$.

In the case when the medium is highly viscous ($\gamma \gg 1$), or the density ρ is very small, we have $\varepsilon \to 0$ and the formal "limit" of this equation will be the following first order equation

$$u_t = \Delta u.$$

Let us mention some works dedicated to the study of singularly perturbed Cauchy problems for differential equations of second order in Hilbert spaces. In [2], [3], [4], [5], [7], [8], [9], the behaviour of the solutions u_ε to the abstract linear Cauchy problem (P_ε) has been studied as $\varepsilon \mapsto 0$ in the case when A_0 and A are positive operators, $B = 0$ or B is an linear integrodifferential operator. All results from these papers were obtained using the theory of semigroups of linear operators.

Our approach is based on two key points. The first one is the relationship between the solutions of the problems (P_ε) and (P_0). The second key point consists in obtaining a priori estimates for the solutions of the problems (P_ε), estimates which are uniform with respect to small parameter ε.

2 Preliminaries

The goal of this section is to remind the notations and main assertions which will be used in that follows.

Let $k \in N^*$, $1 \le p \le +\infty$, $(a,b) \subset (-\infty, +\infty)$ and let X be the Banach space. We denote by $W^{k,p}(a,b;X)$ the Banach space of all vectorial distributions $u \in D'(a,b;X)$, $u^{(j)} \in L^p(a,b;X)$, $j = 0, 1, \dots, k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)} \right)^{1/p}$$

for $p \in [1, \infty)$ and

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0 \le j \le k} \|u^{(j)}\|_{L^\infty(a,b;X)}$$

for $p = \infty$.

In the particular case $p = 2$, we denote $W^{k,2}(a,b;X) = H^k(a,b;X)$. If X is a Hilbert space, then $H^k(a,b;X)$ is also a Hilbert space with the inner product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \Big(u^{(j)}(t), v^{(j)}(t)\Big)_X \, dt.$$

For each arbitrary but fixed $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$, we define the Banach space

$$W_s^{k,p}(a,b;H) = \{f : (a,b) \to H; f^{(l)}(\cdot)e^{-st} \in L^p(a,b;X),\, l = 0,\dots,k\},$$

with the norm

$$\|f\|_{W_s^{k,p}(a,b;X)} = \|fe^{-st}\|_{W^{k,p}(a,b;X)}.$$

Theorem 1. *Let $p \in [1,\infty]$ and X be a reflexive Banach space. Then the embedding $W^{1,p}(0,T;\,X) \hookrightarrow C([0,T];X)$ is continuous, i.e., there exists $C(T,p) > 0$ such that, for each $f \in W^{1,p}(0,T;\,X)$, we have*

$$\|f\|_{C([0,T];X)} \leq C(T,p)\, \|f\|_{W^{1,p}(0,T;\,X)}.$$

Theorem 2. *Let $k \in \mathbb{N}$, $p \in [1,\infty]$ and let X be a Banach space. Then there exists $C(k,p,T) > 0$ such that, for every $f \in W^{k,p}(0,\,T;\,X)$, there exists an extension $\tilde{f} \in W^{k,p}(0,\infty;X)$ of f satisfying*

$$\|\tilde{f}\|_{W^{k,p}(0,\infty;\,X)} \leq C(k,p,T)\, \|f\|_{W^{k,p}(0,T;X)}.$$

Theorem 3. *Let X be a reflexive Banach space. Let $f : (0,T) \to X$ and let $f_h(t) = h^{-1}\left(f(t+h) - f(t)\right)$, $t, t+h \in (0,T)$.*

(*i*) *If $1 \leq p \leq +\infty$ and for each $(a.b) \subseteq (0,T)$ $f \in W^{1,p}(a,b;X)$, then*

$$\|f_h\|_{L^p(a,b;X)} \leq \|f\|_{W^{1,p}(a,b;X)},\; 0 < |h| < \min\{a/2, (T-b)/2\}.$$

(*ii*) *If $1 < p < +\infty$, $f \in L^p(a,b;X)$ and there exists $C > 0$ such that*

$$\|f_h\|_{L^p(a,b;X)} \leq C, \quad 0 < |h| < \min\{a/2, (T-b)/2\},$$

then $f \in W^{1,p}(a,b;\,X)$ and

$$\|f\|_{W^{1,p}(a,b;X)} \leq C.$$

Theorem 4. *Let H be a real Hilbert space, and let $A : D(A) \subset H \to H$ be a linear self-adjoint positive operator. If $u \in W^{1,2}(0,T;H)$ such that $u(t) \in D(A)$ a.e. for $t \in [a,b] \subseteq [0,T]$ and $Au \in L^2(0,T;H)$, then the function $t \to (Au(t), u(t))$ is absolutely continuous on $[a,b]$ and*

$$\frac{d}{dt}(Au(t), u(t)) = 2(Au(t), u'(t)), \quad a.e. \quad t \in [a,b].$$

Definition 1. The operator $A : D(A) \subset H \to H$ is called *monotone* if

$$(Au_1 - Au_2, u_1 - u_2) \geq 0, \quad \forall u_1, u_2 \in D(A).$$

The operator A is called *maximal monotone* if it is monotone and A does not have (possible multivalued) monotone extensions in H.

Theorem 5. [1] *Let $A : D(A) \subset H \to H$ be a monotone operator in H. A is maximal monotone if and only if for every $\lambda > 0$ (equivalently for some $\lambda > 0$), $R(I + \lambda A) = H$.*

Theorem 6. [1] *The linear monotone operator $A : D(A) \subset H \to H$ is maximal monotone if and only if A is closed and $(A^*u, u) \geq 0$, $\forall u \in D(A^*)$, where A^* is the adjoint operator to A.*

For a maximal monotone operator $A : D(A) \subset H \to H$ and $\lambda > 0$, we denote by J_λ its resolvent $J_\lambda = (I + \lambda A)^{-1}$, and by $A_\lambda = \lambda^{-1}(I - J_\lambda)$ the Yosida approximation.

Theorem 7. *Let $A : D(A) \subset H \to H$ be maximal monotone operator. Then for every $\lambda > 0$:*

(*i*) *J_λ is lipschitzian on H with the constant 1;*

(*ii*) *$A_\lambda x = AJ_\lambda x, \quad \forall x \in H \quad$ and $A_\lambda x = J_\lambda Ax, \quad \forall x \in D(A)$;*

(*iii*) *A_λ is a monotone and lipschitzian operator on H with the constant λ^{-1} ;*

(*iv*) *$|A_\lambda x| \leq |Ax|, \quad \forall x \in D(A)$;*

(*v*) *$lim_{\lambda \to 0} A_\lambda x = Ax, \quad \forall x \in D(A)$;*

(*vi*) *$|A_\lambda x|^2 \leq (Ax, A_\lambda x), \quad \forall x \in D(A)$.*

Definition 2. The function $u : [a, b] \to H$ is called *strong solution* to the Cauchy problem

$$\begin{cases} u'(t) + Au(t) = f(t), \quad t \in (a,b), \\ u(a) = u_0 \end{cases} \tag{2.1}$$

if u is absolutely continuous on $[a, b]$, $u' \in L^1(a, b; H)$, $u(t) \in D(A)$ a.e. for $t \in (a, b)$, $u(t)$ satisfies the first equality in (2.1) a.e. for $t \in (0, T)$ and $u(a) = u_0$.

Theorem 8. [1] *Let $A : D(A) \subset H \to H$ such that $A + \omega I$ is maximal monotone. If $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$ then there exists a unique strong solution $u \in W^{1,\infty}(0, T; H)$ to the problem*

$$\begin{cases} u'(t) + Au(t) = f(t), \quad t \in (0,T), \\ u(0) = u_0 \end{cases}$$

and

$$|u(t)| + \left(\int_0^t e^{\gamma\,(t-s)} \left((A + \omega I)u(s), u(s) \right) ds \right)^{1/2}$$
$$\leq e^{\omega\, t/2} \left(|u_0| + \int_0^t e^{-\omega\, s/2} \, |f(s)| \, ds \right), \quad \forall t \in [0, T],$$
$$\left| \frac{d^+u}{dt}(t) \right| \leq e^{\omega t} \, |f(0) - Au_0| + \int_0^t e^{\omega(t-s)} \left| \frac{df}{ds}(s) \right| ds, \quad \forall t \in [0, T).$$

Lemma 1. [10] *Let $\psi \in L^1(a, b)$ $(-\infty < a < b < \infty)$ with $\psi \geq 0$ a.e. on (a,b) and c be a fixed real constant. If $h \in C[a, b]$ verifies*

$$\frac{1}{2} h^2(t) \leq \frac{1}{2} c^2 + \int_a^t \psi(s) h(s) ds, \quad \forall t \in [a, b],$$

then

$$h(t) \leq |c| + \int_a^t \psi(s) ds, \quad \forall t \in [a, b].$$

3 Existence of strong solutions to both (P_ε) and (P_0)

In this section we will study the solvability of problems (P_ε) and (P_0) and also the regularity of their solutions.

The following two theorems were inspired by [1].

Theorem 9. *Let $T > 0$ and let us assume that A_0 satisfies the condition* (**H1**). *If $u_0 \in D(A_0)$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique strong solution $v \in W^{1,\infty}(0,T;H)$ to the problem (P_0). Moreover, v satisfies*

$$|v(t)| + \left(\int_0^t \left| A_0^{1/2} u(s) \right| ds \right)^{1/2} \leq |u_0| + \int_0^t |f(s)|\, ds, \quad \forall t \in [0,T],$$

$$\left| v'(t) \right| \leq |A_0 u_0 - f(0)| + \int_0^t \left| f'(s) \right| ds, \quad \forall t \in [0,T].$$

Theorem 10. *Let $T > 0$. Let us assume that $A : D(A) \subset H \to H$ is linear self-adjoint and positive. If $u_0 \in D(A)$, $u_1 \in H$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique function $u : [0,T] \to H$ such that:*

$u \in W^{2,\infty}(0,T;H), \quad A^{1/2}u \in W^{1,\infty}(0,T;H), \quad Au \in L^\infty(0,T;H),$

$A^{1/2}u$ *and* u' *are differentiable from to the right in* H *for every* $t \in [0,T)$ *and*

$$\frac{d^+}{dt}\frac{du}{dt}(t) + \frac{du}{dt}(t) + Au(t) = f(t), \quad t \in [0,T), \tag{3.1}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{3.2}$$

In what follows this function will be called *the strong solution* to the problem (3.1), (3.2).

Proof. Let us denote by $\mathcal{H} = D(A^{1/2}) \times H$ which, endowed with the inner product

$$(U_1, U_2)_{\mathcal{H}} = (A^{1/2}u_1, A^{1/2}u_2) + (v_1, v_2), \quad U_i = (u_i; v_i) \in \mathcal{H}, \quad i = 1, 2,$$

is the real Hilbert space. Let us further denote by $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subseteq \mathcal{H} \to \mathcal{H}$, the operator defined by

$$D(\mathcal{L}) = D(A) \times H, \quad \mathcal{L}U = (-v, Au + v), \quad \forall U = (u; v) \in D(\mathcal{L}).$$

As

$$(\mathcal{L}U, U)_{\mathcal{H}} = -(Av, u) + (Au + v, v) = |v|^2 \geq 0, \quad \forall U \in D(\mathcal{L}),$$

it follows that $\mathcal{L}$ is monotone. Now we are going to show that it is maximal monotone. To this aim, let us consider the equation $(\lambda I + \mathcal{L})\, U = F$, $\lambda > 0$, where $F = (f, g) \in \mathcal{H}$ and $U = (u, v) \in D(\mathcal{L})$, which is equivalent to the system

$$\begin{cases} \lambda u - v = f \\ \lambda v + Au + v = g, \end{cases}$$

i.e.

$$\begin{cases} \lambda u - v = f \\ \lambda(\lambda+1)u + Au = g_1, \end{cases} \tag{3.3}$$

where $g_1 = g + (\lambda + 1)f$.

As A is a positive self-adjoint operator, therefore using Theorem 6, we can infer that A is a maximal monotone operator. Due to Theorem 5, we have that

$$\forall \beta > 0 \quad D((\beta I + A)^{-1}) = H, \quad R((\beta I + A)^{-1}) \subseteq D(A).$$

Therefore (3.3) is equivalent to the system

$$\begin{cases} \lambda u - v = f \\ u = (\beta I + A)^{-1} g_1, \end{cases} \tag{3.4}$$

with $\beta = \lambda(\lambda + 1)$. Hence, if $f \in D(A^{1/2})$ and $g \in H$, it follows that $u = (\beta I + A)^{-1} g_1 \in D(A)$. From the first equation in (3.4), we deduce that $v = \lambda u - f \in D(A^{1/2})$. So, for every $F \in \mathcal{H}$ there exists a unique solution $U \in D(\mathcal{L})$ to the equation $(\lambda I + \mathcal{L})\, U = F$. So, $R(\lambda I + \mathcal{L}) = \mathcal{H}$ and, by Theorem 5, the operator $\mathcal{L}$ is maximal monotone. By Theorem 8, the problem

$$\begin{cases} U'(t) + \mathcal{L}U(t) = F(t), \quad t \in (0,T), \\ U(0) = U_0, \end{cases} \tag{P.U}$$

where $U(t) = (u(t); v(t))$, $U_0 = (u_0, u_1)$, $F(t) = (0, f(t))$ has a unique strong solution $U = (u, v) \in W^{1,\infty}(0, T; \mathcal{H})$ which implies that $A^{1/2}u, v \in W^{1,\infty}(0, T; H)$. As the equation in $(P.U)$ is equivalent to the system

$$\begin{cases} u'(t) - v(t) = 0 \\ v'(t) + Au(t) + v(t) = f(t), \end{cases}$$

it follows that u satisfies (3.1) and (3.2). Thus, (3.1), (3.2) has a unique strong solution $u \in W^{2,\infty}(0, T; H)$.

Finally, we have $A^{1/2}u \in W^{1,\infty}(0, T; H)$ and $Au \in L^\infty(0, T; H)$ and this completes the proof. $\square$

4 A priori estimates for solutions to the problem (P_ε)

The goal of this section is to establish some *a priori* estimations for solutions to (P_ε) which are uniform relative to the small parameter ε.

Consider the following problem:

$$\begin{cases} \varepsilon\left(u_\varepsilon''(t)+A_1u_\varepsilon(t)\right)+u_\varepsilon'(t)+A_0u_\varepsilon(t)=f(t), \quad t\in(0,T),\\ u_\varepsilon(0)=u_0, \quad u_\varepsilon'(0)=u_1. \end{cases} \tag{4.1}$$

Lemma 2. *Let $T>0$. Suppose that, for each $\varepsilon\in(0,1)$, the operator $A(\varepsilon)=(\varepsilon A_1+A_0):D\left(A(\varepsilon)\right)\subseteq H\to H$ is self-adjoint and satisfies*

$$(A(\varepsilon)u,u)\geq\omega\,|u|^2, \quad \forall u\in D\left(A(\varepsilon)\right), \quad \omega>0, \quad \varepsilon\in(0,1]. \tag{4.2}$$

If $f\in W^{1,1}(0,T;H)$, $u_0\in D\left(A(\varepsilon)\right)$, $u_1\in H$, then the unique strong solution, u_ε, of the problem (4.1) *satisfies*

$$\|A^{1/2}(\varepsilon)u_\varepsilon\|_{C([0,t];\,H)}+\|u_\varepsilon'\|_{L^2(0,t;\,H)}\leq C(\omega)\,M(t), \tag{4.3}$$

for each $t\in[0,T]$ and each $\varepsilon\in(0,1/2]$. If, in addition, $u_1\in D\left(A^{1/2}(\varepsilon)\right)$, then

$$\|u_\varepsilon'\|_{C([0,t];\,H)}+\|A^{1/2}(\varepsilon)\,u_\varepsilon'\|_{L^2(0,t;\,H)}\leq C(\omega)\,M_1(t), \tag{4.4}$$

for each $t\in[0,T]$, and each $\varepsilon\in(0,1]$, and

$$\|A(\varepsilon)u_\varepsilon\|_{L^\infty(0,t;\,H)}\leq C(\omega)M_1(t), \quad \forall t\in[0,T], \quad \forall\varepsilon\in(0,1], \tag{4.5}$$

where

$$M(t)=M(t,u_0,u_1,f)=\left|A^{1/2}(\varepsilon)u_0\right|+|u_1|+\|f\|_{W^{1,1}(0,t;\,H)}+|f(0)|,$$

$$M_1(t)=M_1(t,u_0,u_1,f)=\left|A^{1/2}(\varepsilon)u_1\right|+|A(\varepsilon)u_0|+\|f\|_{W^{1,1}(0,t;\,H)}+|f(0)|.$$

Proof. We begin with the proof of (4.3). Let us denote by

$$E(u,t)=\varepsilon\,\left(u'(t),u(t)\right)+\int_0^t\,(A(\varepsilon)u(\tau),u(\tau))\,d\tau+\frac{1}{2}\,|u(t)|^2$$

$$+\varepsilon \int_0^t \left|u'(\tau)\right|^2 \, d\tau + \varepsilon^2 \left|u'(t)\right|^2 + \varepsilon \, (A(\varepsilon)u(t), u(t)) \, .$$

For every solution, u_ε, of (4.1), by direct computation, we obtain

$$\frac{d}{dt} E(u_\varepsilon, t) = \big(f(t), u_\varepsilon(t) + 2\varepsilon u_\varepsilon'(t)\big) \, , \quad \text{a.e.} \quad t \in (0, T).$$

As

$$E(u_\varepsilon, t) \geq 0, \quad \left|u_\varepsilon(t) + 2\varepsilon u_\varepsilon'(t)\right| \leq 2 \, (E(u_\varepsilon, t))^{1/2} \, ,$$

for each $t \in [0, T]$, and each $\varepsilon \in (0, 1]$, it follows that

$$\frac{d}{dt} E(u_\varepsilon, t) \leq 2 \, |f(t)| \; (E(u_\varepsilon, t))^{1/2} \, , \quad \text{a.e.} \quad \forall t \in (0, T).$$

Integrating the last inequality, we obtain

$$\frac{1}{2} E(u_\varepsilon, t) \leq \frac{1}{2} E(u_\varepsilon, 0) + \int_0^t |f(\tau)| \, (E(u_\varepsilon, \tau))^{1/2} \, d\tau, \quad \forall t \in [0, T].$$

Applying Lemma 1 to the last inequality, we get

$$(E(u_\varepsilon, t))^{1/2} \leq (E(u_\varepsilon, 0))^{1/2} + \int_0^t |f(\tau)| \, d\tau, \quad \forall t \in [0, T],$$

from which we deduce

$$\|u_\varepsilon\|_{C([0,\,t];\,H)} + \|A^{1/2}(\varepsilon)\, u_\varepsilon\|_{L^2(0,\,t;\,H)} \leq C(\omega) M(t), \tag{4.6}$$

for each $t \in [0, T]$ and each $\forall \varepsilon \in (0, 1]$. Let now

$$\mathcal{E}(u, t) = \varepsilon |u'(t)|^2 + |u(t)|^2 + (A(\varepsilon)u(t), u(t)) + 2(1 - \varepsilon) \int_0^t |u'(s)|^2 ds$$

$$+2\varepsilon \, \big(u(t), u'(t)\big) + 2 \int_0^t (A(\varepsilon)u(s), u(s)) \, ds.$$

Then, for every strong solution u_ε to the problem (4.1), we have

$$\frac{d}{dt} \mathcal{E}(u_\varepsilon, t) = 2 \, \big(f(t), u_\varepsilon(t) + u_\varepsilon'(t)\big) \, , \quad \text{a.e. } t \in (0, T),$$

and thus

$$\mathcal{E}(u_\varepsilon,t)=\mathcal{E}(u_\varepsilon,0)+2\,(u_\varepsilon,f(t))-2\,(u_0,f(0))$$
$$+2\int_0^t \big(f(s)-f'(s),u_\varepsilon(s)\big)\,ds,\quad \forall t\in[0,T]. \tag{4.7}$$

Since

$$\mathcal{E}(u_\varepsilon,0)\le C(\omega)\,M^2(t),\quad \forall t\in[0,T],\quad \forall\varepsilon\in(0,1]$$

and, in view of (4.6), we have

$$2\,|(u_\varepsilon,f(t))-(u_0,f(0))|\le C(\omega)\,M^2(t),\quad \forall t\in[0,T],\quad \forall\varepsilon\in(0,1],$$

from (4.7), we get

$$\mathcal{E}(u_\varepsilon,t)\le C(\omega)\,M^2(t),\quad t\in[0,t],\quad \forall\varepsilon\in(0,1],$$

which implies (4.3).

Proof of (4.4). Let $h>0$ such that $t,t+h\in[0,T]$. Denote by $u_{\varepsilon h}(t)=u_\varepsilon(t+h)-u_\varepsilon(t)$, where u_ε is the strong solution to problem (4.1). Then for $u_{\varepsilon h}$ we have the equality

$$\frac{d}{dt}E(u_{\varepsilon h},t)=\big(f_h(t),u_{\varepsilon h}(t)+2\varepsilon\,u'_{\varepsilon h}(t)\big)\quad \text{a.e.}\quad \in(0,T-h).$$

Integrating this equality and applying Lemma 1 and Theorem 3, we obtain

$$\left(E(u_{\varepsilon h},t)\right)^{1/2}\le\left(E(u_{\varepsilon h},0)\right)^{1/2}+\int_0^t\left|f'(\tau)\right|\,d\tau,\quad \forall t\in[0,T-h].$$

As $u'(0)=u_1$ and

$$\lim_{h\downarrow 0}\varepsilon\left|h^{-1}\,u'_h(0)\right|=\left|f(0)-u_1-A(\varepsilon)u_0\right|,$$

$$\lim_{h\downarrow 0}h^{-1}\left|A^{1/2}(\varepsilon)u_{\varepsilon h}(0)\right|=\left|A^{1/2}(\varepsilon)u_1\right|,$$

dividing the last equality by h and passing to the limit as $h\to 0$, we get (4.4).

Proof of (4.5). Let $A_\lambda(\varepsilon)$ be the Yosida approximation of the operator $A(\varepsilon)$. Let

$$E_1(u,t)=\varepsilon\left(A_\lambda(\varepsilon)u'(t),u'(t)\right)+\left(A_\lambda(\varepsilon)u(t),u(t)\right)$$

$$+\left(A_\lambda(\varepsilon)u(t),A(\varepsilon)u(t)\right)+2\varepsilon\left(A_\lambda(\varepsilon)u(t),u'(t)\right)$$

$$+2(1-\varepsilon)\int_0^t\left(A_\lambda(\varepsilon)u'(s),u'(s)\right)\,ds+2\int_0^t\left(A_\lambda(\varepsilon)u(\tau),A(\varepsilon)u(s)\right)ds.$$

Then every strong solution, u_ε, of the problem (4.1) satisfies

$$\frac{d}{dt}E_1(u_\varepsilon,t)=2\left(f(t),\mathcal{A}_\lambda u_\varepsilon(t)+\mathcal{A}_\lambda u_\varepsilon'(t)\right),\quad \text{a.e.}\quad t\in(0,T).$$

Integrating this equality, we obtain

$$E_1(u_\varepsilon,t)=E_1(u_\varepsilon,0)+I_1(t,\varepsilon)+I_2(t,\varepsilon),\quad \forall t\in[0,T],\tag{4.8}$$

where

$$I_1(t,\varepsilon)=2\left(f(t),A_\lambda(\varepsilon)u_\varepsilon(t)\right)-2\left(f(0),A_\lambda(\varepsilon)u_0\right),$$

$$I_2(t,\varepsilon)=2\int_0^t\left(f(s)-f'(s),A_\lambda(\varepsilon)u_\varepsilon(s)\right)\,ds.$$

Let us evaluate $I_1(t,\varepsilon),I_2(t,\varepsilon)$. Using (iv), (vi) in Theorem 7, we get

$$|I_1(t,\varepsilon)|\leq\frac{1}{2}\left|A_\lambda(\varepsilon)u_\varepsilon(t)\right|^2+2|f(t)|^2+|f(0)|^2+\left|A_\lambda(\varepsilon)u_0\right|^2$$

$$\leq\frac{1}{2}\left(A_\lambda(\varepsilon)u_\varepsilon(t),A(\varepsilon)u_\varepsilon(t)\right)+C(\omega)\,M_1^2(t),\quad \forall t\in[0,T].\tag{4.9}$$

As $(A_\lambda(\varepsilon)u,u)\geq 0$, $\forall u\in H$, it follows that

$$\left(A_\lambda(\varepsilon)u,v\right)^2\leq\left(A_\lambda(\varepsilon)u,u\right)\,\left(A_\lambda(\varepsilon)v,v\right),\quad \forall u,v\in H.$$

Therefore, due to (vi) in Theorem 7, we get

$$\varepsilon\left(A_\lambda(\varepsilon)u_\varepsilon'(t),u_\varepsilon'(t)\right)+\left(A_\lambda(\varepsilon)u_\varepsilon(t),u_\varepsilon(t)\right)+\left(A_\lambda(\varepsilon)u_\varepsilon(t),A(\varepsilon)u_\varepsilon(t)\right)$$

$$+2\varepsilon\left(A_\lambda(\varepsilon)u_\varepsilon(t),u_\varepsilon'(t)\right)=(1-\varepsilon)\,\left(A_\lambda(\varepsilon)u_\varepsilon(t),u_\varepsilon(t)\right)$$

$$+\varepsilon\left(A_\lambda(u_\varepsilon(t)+u_\varepsilon'(t)),(u_\varepsilon(t)+u_\varepsilon'(t)\right)+\left(A_\lambda(\varepsilon)u_\varepsilon(t),A(\varepsilon)u_\varepsilon(t)\right)$$

$$\geq\left(A_\lambda(\varepsilon)u_\varepsilon(t),A(\varepsilon)u_\varepsilon(t)\right)\geq\left|A_\lambda(\varepsilon)u_\varepsilon(t)\right|^2,\quad \forall\varepsilon\in(0,1].$$

As

$$E_1(u_\varepsilon,t)\geq 0,\quad |A_\lambda(\varepsilon)u_\varepsilon|\leq E_1^{1/2}(u_\varepsilon,t),\quad \forall t\in[0,T],\quad \forall\varepsilon\in(0,1],$$

we have

$$|I_2(t,\varepsilon)| \leq 2\int_0^t \big(|f(s)| + |f'(s)|\big)\ E_1^{1/2}(u_\varepsilon, s)\,ds, \quad \forall t \in [0,T]. \tag{4.10}$$

Due to (vi) in Theorem 7, we get

$$E_1(u_\varepsilon, 0) \leq C(\omega)\left(|A(\varepsilon)u_0|^2 + |A^{1/2}(\varepsilon)u_1|^2\right), \quad \forall \varepsilon \in (0,1]. \tag{4.11}$$

Using (4.9), (4.10) and (4.11), from (4.8), we obtain

$$E_1(u_\varepsilon, t) \leq C(\omega)\,M_1^2(t)$$

$$+2\int_0^t \big(|f(s)| + |f'(s)|\big)\ E_1^{1/2}(u_\varepsilon, s)\,ds, \tag{4.12}$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,1]$.

Applying Lemma 1 to (4.12), we deduce

$$E_1^{1/2}(u_\varepsilon, t) \leq C(\omega)\,M_1(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1],$$

from which it follows that

$$(A_\lambda(\varepsilon)u_\varepsilon(t), A(\varepsilon)u_\varepsilon(t)) \leq C(\omega)\,M_1^2(t)\forall \quad t \in [0,T], \quad \forall \varepsilon \in (0,1].$$

Finally, passing to the limit in the last inequality as $\lambda \to 0$ and using (v) in Theorem 7, we get (4.5) and this completes the proof. □

Let u_ε be a strong solution of the problem (4.1) and let us denote by

$$z_\varepsilon(t) = u'_\varepsilon(t) + \alpha e^{-t/\varepsilon}, \quad \alpha = f(0) - u_1 - A(\varepsilon)u_0. \tag{4.13}$$

Lemma 3. *Let $T > 0$ and let us assume that, for each $\varepsilon \in (0,1)$, the operator $A(\varepsilon) = \varepsilon A_1 + A_0$ is self-adjoint and satisfies* (4.2). *If u_1, $f(0) - A(\varepsilon)u_0 \in D\left(A(\varepsilon)\right)$ and $f \in W^{2,1}(0,T;H)$, then there exist $C(\omega) > 0$, such that the function z_ε, defined by* (4.13)*, satisfies*

$$\|A^{1/2}(\varepsilon)z_\varepsilon\|_{C([0,\,t];\,H)} + \|z'_\varepsilon\|_{C([0,\,t];\,H)} + \left\|A^{1/2}(\varepsilon)z'_\varepsilon\right\|_{L^2(0,\,t;\,H)}$$

$$\leq C(\omega)\,M_2(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1], \tag{4.14}$$

where

$$M_2(t) = |A(\varepsilon)f(0) - A^2(\varepsilon)u_0| + \|f\|_{W^{2,1}(0,t;H)} + |A(\varepsilon)u_1| + |f'(0)|.$$

Proof. If u_1, $f(0) - A(\varepsilon)u_0 \in D\,(A(\varepsilon))$ and $f \in W^{2,1}(0,T;H)$, then, due to Theorem 10, z_ε is the strong solution of the problem

$$\begin{cases} \varepsilon z_\varepsilon''(t) + z_\varepsilon'(t) + A(\varepsilon)z_\varepsilon(t) = \mathcal{F}(t,\varepsilon), \quad \text{a.e.} \quad t \in (0,T), \\ z_\varepsilon(0) = f(0) - A(\varepsilon)u_0, \quad z_\varepsilon'(0) = 0, \end{cases}$$

where

$$\mathcal{F}(t,\varepsilon) = f\,'(t) + e^{-t/\varepsilon}\,A(\varepsilon)\alpha.$$

Finally, let us observe that z_ε satisfies $A^{1/2}(\varepsilon)z_\varepsilon \in W^{1,\infty}(0,T;H)$, $z_\varepsilon \in W^{2,\infty}(0,T;H)$ and $A(\varepsilon)z_\varepsilon \in L^\infty(0,T;H)$. Therefore, (4.14) follows from Lemma 2 and the proof is complete. □

5 The relationship between the solution of (P_ε) and (P_0)

Now we are going to establish the relationship between the solution to the problem (P_ε) and the corresponding solution to the problem (P_0). This relationship was inspired by [6]. To this end, we begin by defining the transformation kernel which realizes this relationship.

Namely, for $\varepsilon > 0$, let us denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\,\varepsilon\,\sqrt{\pi}}\left(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\right),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\left\{\frac{3t-2\tau}{4\varepsilon}\right\}\lambda\left(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\right),$$

$$K_2(t,\tau,\varepsilon) = \exp\left\{\frac{3t+6\tau}{4\varepsilon}\right\}\lambda\left(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\right),$$

$$K_3(t,\tau,\varepsilon) = \exp\left\{\frac{\tau}{\varepsilon}\right\}\lambda\left(\frac{t+\tau}{2\sqrt{\varepsilon t}}\right), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}\,d\eta.$$

The properties of the kernel $K(t,\tau,\varepsilon)$ are collected in the next lemma.

Lemma 4. [11]. *The function $K(t,\tau,\varepsilon)$ has the following properties:*

(*i*) $K \in C([0,\infty) \times [0,\infty)) \cap C^2((0,\infty) \times (0,\infty))$;

(*ii*) $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t > 0, \forall \tau > 0$;

(*iii*) $\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \geq 0\,;$

(*iv*) $K(0,\tau,\varepsilon) = \dfrac{1}{2\varepsilon}\exp\left\{-\dfrac{\tau}{2\varepsilon}\right\}, \quad \forall \tau \geq 0\,;$

(*v*) *For every* $t > 0$ *and every* $q, s \in \mathbb{N}$, *there exist* $C_1(q,s,t,\varepsilon) > 0$ *and* $C_2(q,s,t) > 0$ *such that*

$$|\partial_t^s \partial_\tau^q K(t,\tau,\varepsilon)| \leq C_1(q,s,t,\varepsilon)\exp\{-C_2(q,s,t)\tau/\varepsilon\}, \quad \forall \tau > 0\,;$$

Moreover, for every $\gamma \in \mathbb{R}$, *there exist* $C_1 > 0$, $C_2 > 0$ *and* $\varepsilon_0 > 0$, *depending on* γ, *such that*:

$$\int_0^\infty e^{\gamma\,\tau}\;|K_t(t,\tau,\varepsilon)|\;d\tau \leq C_1\,\varepsilon^{-1}\,e^{C_2 t}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon_0],$$

$$\int_0^\infty e^{\gamma\,\tau}\;|K_\tau(t,\tau,\varepsilon)|\;d\tau \leq C_1\,\varepsilon^{-1}\,e^{C_2 t} \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon_0],$$

$$\int_0^\infty e^{\gamma\,\tau}\;|K_{\tau\,\tau}(t,\tau,\varepsilon)|\;d\tau \leq C_1\,\varepsilon^{-2}\,e^{C_2 t}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon_0]\,;$$

(*vi*) $K(t,\tau,\varepsilon) > 0, \quad \forall t \geq 0, \quad \forall \tau \geq 0\,;$

(*vii*) *For every continuous* $\varphi : [0,\infty) \to H$, *with* $|\varphi(t)| \leq M\exp\{\gamma\,t\}$, *we have*:

$$\lim_{t\to 0}\left\|\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau - \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau\right|_H = 0,$$

for every $\varepsilon \in \left(0,(2\,\gamma)^{-1}\right)$;

(*viii*)

$$\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad \forall t \geq 0.$$

(*ix*) *For every* $\gamma > 0$ *and* $q \in [0,1]$, *there exist* $C_1 > 0$, $C_2 > 0$ *and* $\varepsilon_0 > 0$, *depending on* γ *and on* q, *such that* :

$$\int_0^\infty K(t,\tau,\varepsilon)\,e^{\gamma\tau}|t-\tau|^q\,d\tau \leq C_1\,e^{C_2 t}\,\varepsilon^{q/2}, \quad \forall t > 0, \quad \forall \varepsilon \in (0,\varepsilon_0].$$

If $\gamma \leq 0$ *and* $q \in [0,1]$, *then*

$$\int_0^\infty K(t,\tau,\varepsilon)\,e^{\gamma\tau}\,|t-\tau|^q\,d\tau \leq C\,\varepsilon^{q/2}\,\left(1+\sqrt{t}\right)^q,\ \forall t \geq 0, \forall \varepsilon \in (0,1]\,;$$

(*x*) *Let* $p \in (1, \infty]$ *and* $f : [0, \infty) \to H$, $f \in W^{1,p}_\gamma(0,\infty;H)$. *If* $\gamma > 0$, *then there exist* $C_1 > 0$, $C_2 > 0$ *and* ε_0 *depending on* γ *and* p, *such that*

$$\Big\| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \Big\|_H$$
$$\le C_1\, e^{C_2 t}\, \|f'\|_{L^p_\gamma(0,\infty;H)}\, \varepsilon^{(p-1)/2p}, \ \forall t \ge 0, \ \forall \varepsilon \in (0, \varepsilon_0].$$

If $\gamma \le 0$, *then*

$$\Big\| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \Big\|_H$$
$$\le C(\gamma,p)\, \|f'\|_{L^p_\gamma(0,\infty;H)} \left(1 + \sqrt{t}\right)^{\frac{p-1}{p}} \varepsilon^{(p-1)/2p}, \ \forall t \ge 0, \ \forall \varepsilon \in (0,1].$$

(*xi*) *For every* $q > 0$ *and* $\alpha \ge 0$, *there exists* $C(q,\alpha) > 0$ *such that*

$$\int_0^t \int_0^\infty K(\tau,\theta,\varepsilon)\, e^{-q\,\theta/\varepsilon}\, |\tau - \theta|^\alpha\, d\theta\, d\tau \le C(q,\alpha)\, \varepsilon^{1+\alpha},$$

for each $t \ge 0$, *and each* $\varepsilon > 0$.

Now we are ready to establish the relationship between the solution of (P_ε) and the solution of (P_0).

Theorem 11. *Suppose that* $A(\varepsilon)$ *satisfies* **(H1)**. *Let* $f \in L^\infty_c(0,\infty;H)$ *and let* $u_\varepsilon \in W^{2,\infty}_c(0,\infty;H)$ *be the strong solution of the problem* (4.1), *with* $Au_\varepsilon \in L^\infty_c(0,\infty;H)$, *for some* $c \ge 0$. *Then the function* w_ε, *defined by*

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\, u_\varepsilon(\tau)\, d\tau,$$

is the strong solution of the problem

$$\begin{cases} w'_\varepsilon(t) + A(\varepsilon) w_\varepsilon(t) = F_0(t,\varepsilon), \quad t > 0, \\ w_\varepsilon(0) = \varphi_\varepsilon, \end{cases} \tag{5.1}$$

where

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau} u_\varepsilon(2\varepsilon\tau) d\tau, \quad F_0(t,\varepsilon) = f_0(t,\varepsilon) u_1 + \int_0^\infty K(t,\tau,\varepsilon)\, f(\tau)\, d\tau,$$

$$f_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}} \left[2 \exp\left\{ \frac{3t}{4\varepsilon} \right\} \lambda\left(\sqrt{\frac{t}{\varepsilon}} \right) - \lambda\left(\frac{1}{2} \sqrt{\frac{t}{\varepsilon}} \right) \right].$$

Proof. Integrating by parts and using (i),(ii) and (iii) in Lemma 4, we get

$$(\omega_\varepsilon'(t), \eta) = \left(\int_0^\infty K_t(t, \tau, \varepsilon) u_\varepsilon(\tau) d\tau, \eta \right)$$

$$= \left(\int_0^\infty \left[\varepsilon K_{\tau\tau}(t, \tau, \varepsilon) - K_\tau(t, \tau, \varepsilon) \right] u_\varepsilon(\tau) d\tau, \eta \right)$$

$$= - \left(\left[\varepsilon K_\tau(t, 0, \varepsilon) - K(t, 0, \varepsilon) \right] u_\varepsilon(0), \eta \right) + \left(\varepsilon K(t, 0, \varepsilon) u_1, \eta \right)$$

$$+ \left(\int_0^\infty K(t, \tau, \varepsilon) \left(\varepsilon u_\varepsilon''(\tau) + u_\varepsilon'(\tau) \right) d\tau, \eta \right)$$

$$= \left(\varepsilon K(t, 0, \varepsilon) u_1, \eta \right) + \left(\int_0^\infty K(t, \tau, \varepsilon) \left[f(\tau) - A(\varepsilon) u_\varepsilon(\tau) \right] d\tau, \eta \right)$$

$$= \left(\varepsilon K(t, 0, \varepsilon) u_1 + \int_0^\infty K(t, \tau, \varepsilon) f(\tau) d\tau, \eta \right) - \left(A(\varepsilon) w_\varepsilon(t), \eta \right)$$

$$= \left(f_0(t, \varepsilon) u_1 + \int_0^\infty K(t, \tau, \varepsilon) f(\tau) d\tau, \eta \right) - \left(A(\varepsilon) w_\varepsilon(t), \eta \right),$$

for each $\eta \in D(A(\varepsilon))$. Thus

$$\left(w_\varepsilon'(t) + A w_\varepsilon(t) - F_0(t, \varepsilon), \eta \right) = 0, \quad \forall \eta \in D(A(\varepsilon)),\ a.e.\ t > 0.$$

Let us observe that $F_0(t, \varepsilon) \in L^\infty_{c_1}(0, \infty; H)$ and from (v) in Lemma 4, we conclude that $w_\varepsilon' \in L^\infty_{c_1}(0, \infty; H)$ (with some $c_1 > 0$), which implies that $A(\varepsilon) w_\varepsilon \in L^\infty_{c_1}(0, \infty; H)$. Since $\overline{D(A)} = H$, it follows that $w_\varepsilon(t)$ satisfies the first equation in (5.1) a.e. $t > 0$.

As the initial condition is a simple consequence of (iv) and (vii) in Lemma 4, the proof is complete. □

6 The limit of the solutions of the problem (P_ε) as $\varepsilon \longrightarrow 0$

In this section we will study the behavior of solutions to the problem (P_ε) as $\varepsilon \to 0$.

Theorem 12. *Let $T > 0$ and $p \in (1, \infty]$. Let us assume that the operators A_0 and A_1 satisfy* **(H1)** *and* **(H2)**. *If*

$$u_0, u_{0\,\varepsilon} \in D(A_0), \quad u_{1\,\varepsilon} \in H, \quad f, f_\varepsilon \in W^{1,p}(0,T;H),$$

then there exist $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1) \in (0,1)$ and $C = C(T, p, \omega_0, \omega_1) > 0$ such that

$$\|u_\varepsilon - v\|_{C([0,T];H)} \\ \leq C\, \left(\mathcal{M}_\varepsilon\, \varepsilon^\beta + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.1}$$

for all $\varepsilon \in (0, \varepsilon_0]$, where u_ε and v are the strong solutions of problems (P_ε) and (P_0) respectively,

$$\beta = \min\{1/4, (p-1)/2p\}$$

and

$$\mathcal{M}_\varepsilon = \left|A_0^{1/2} u_{0\,\varepsilon}\right| + |u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,p}(0,T;H)}.$$

If, in addition, $u_{1\,\varepsilon} \in D\left(A_0^{1/2}\right)$, then, for each $\varepsilon \in (0, \varepsilon_0]$, we have

$$\|u_\varepsilon - v\|_{C([0,T];H)} \\ \leq C\, \left(\mathcal{M}_{1\,\varepsilon}\, \varepsilon^{(p-1)/2p} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.2}$$

and

$$\|A_0^{1/2} u_\varepsilon - A_0^{1/2} v\|_{L^2(0,\,T;\,H)} \\ \leq C\, \left(\mathcal{M}_{1\,\varepsilon}\, \varepsilon^\beta + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.3}$$

where $\beta = \min\{1/4, (p-1)/2p\}$ and

$$\mathcal{M}_{1\,\varepsilon} = \left|A_0^{1/2} u_{1\,\varepsilon}\right| + |A_0 u_{0\,\varepsilon}| + |A_1 u_{0\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,p}(0,\,T;\,H)}.$$

Proof. From (**H1**) and (**H2**), it follows that there exists $\gamma = 3\omega_1 > 0$ such that

$$|(A_1 u, v)| \leq |((A_1 + \omega_1\, A_0) u, v)| + \omega_1\, |A_0^{1/2} u|\, |A_0^{1/2} v|$$
$$\leq ((A_1 + \omega_1\, A_0) u, u)^{1/2}\, ((A_1 + \omega_1\, A_0) v, v)^{1/2} + \omega_1\, |A_0^{1/2} u|\, |A_0^{1/2} v|$$
$$\leq (2\omega_1\, (A_0 u, u))^{1/2}\, (2\omega_1\, (A_0 v, v))^{1/2}$$

$$+\omega_1 \left|A_0^{1/2}u\right| \left|A_0^{1/2}v\right| \leq \gamma \left|A_0^{1/2}u\right| \left|A_0^{1/2}v\right|, \quad \forall u, v \in D(A_0). \tag{6.4}$$

If $f_\varepsilon \in W^{l,p}(0,T;H)$ with $p \in (1,\infty]$ and $l \in \mathbb{N}^\star$, then, due to Theorems 1 and 2, we have that $f_\varepsilon \in C([0,T];H)$ and there exists an extension $\tilde{f}_\varepsilon \in W^{l,p}(0,\infty;H)$ such that

$$\|\tilde{f}_\varepsilon\|_{C([0,\infty);H)} + \|\tilde{f}_\varepsilon\|_{W^{l,p}(0,\infty;H)} \leq C(T,p,l)\,\|f_\varepsilon\|_{W^{l,p}(0,T;H)}. \tag{6.5}$$

Let us denote by $\tilde{u}_\varepsilon$ the unique strong solution to the problem (P_ε) and by $\tilde{v}$ the unique strong solution to the problem (P_0), defined on $(0,\infty)$ instead of $(0,T)$, and f_ε by $\tilde{f}_\varepsilon$. From Theorem 10, we have

$$\begin{cases} \tilde{u}_\varepsilon \in W^{2,\infty}(0,T;H),\ A^{1/2}(\varepsilon)\tilde{u}_\varepsilon \in W^{1,\infty}(0,T;H), \\ A(\varepsilon)\tilde{u}_\varepsilon \in L^\infty(0,T;H),\ \forall T \in (0,\infty). \end{cases}$$

From Lemma 2 and (6.4), it follows that

$$\begin{cases} \tilde{u}_\varepsilon \in W^{2,\infty}(0,\infty;H),\ A_0^{1/2}\tilde{u}_\varepsilon \in W^{1,2}(0,\infty;H), \\ A(\varepsilon)\tilde{u}_\varepsilon \in L^\infty(0,\infty;H). \end{cases}$$

Moreover, due to the same lemma and to (6.4) and (6.5), we get

$$\|A_0^{1/2}\tilde{u}_\varepsilon\|_{C([0,t];\,H)} + \|\tilde{u}_\varepsilon'\|_{L^2(0,t;\,H)} \leq C\,\mathcal{M}_\varepsilon, \forall t \geq 0,\ \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.6}$$

If in addition, $u_{1\,\varepsilon} \in D\left(A_0^{1/2}\right)$, then

$$\|\tilde{u}_\varepsilon'\|_{C([0,t];\,H)} + \|A_0^{1/2}\,\tilde{u}_\varepsilon'\|_{L^2(0,t;\,H)} \leq C\,\mathcal{M}_{1\,\varepsilon}, \tag{6.7}$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$.

Proof of (6.1). According to Theorem 4, the function

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau,$$

is the strong solution to the problem

$$\begin{cases} w_\varepsilon'(t) + A(\varepsilon)w_\varepsilon(t) = F(t,\varepsilon), \quad t > 0, \quad \text{în} \quad H, \\ w_\varepsilon(0) = w_0, \end{cases}$$

for $0 < \varepsilon \leq \varepsilon_0$, where

$$\begin{cases} F(t,\varepsilon) = f_0(t,\varepsilon)\, u_{1\,\varepsilon} + \displaystyle\int_0^\infty K(t,\tau,\varepsilon)\, \tilde{f}_\varepsilon(\tau)\, d\tau, \\ f_0(t,\varepsilon) = \dfrac{1}{\sqrt{\pi}} \left[2\exp\left\{\dfrac{3t}{4\varepsilon}\right\} \lambda\left(\sqrt{\dfrac{t}{\varepsilon}}\right) - \lambda\left(\dfrac{1}{2}\sqrt{\dfrac{t}{\varepsilon}}\right)\right], \\ w_0 = \displaystyle\int_0^\infty e^{-\tau}\, \tilde{u}_\varepsilon(2\varepsilon\tau) d\tau. \end{cases}$$

Using Hölder's inequality, (vi), $(viii)$, (ix) (x) in Lemma 4, and (6.6), we obtain

$$\begin{aligned} \|\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\|_H &= \left\| \tilde{u}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\, \tilde{u}_\varepsilon(\tau)\, d\tau \right\|_H \\ &\leq \int_0^\infty K(t,\tau,\varepsilon)\, \|\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\|_H\, d\tau \\ &\leq \int_0^\infty K(t,\tau,\varepsilon) \left\| \int_t^\tau \|\tilde{u}'_\varepsilon(s)\|_H\, ds \right\| d\tau \\ \leq \|\widetilde{u}'_\varepsilon\|_{L^2(0,\,\infty;\,H)} &\int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|^{1/2}\, d\tau \leq C\, \mathcal{M}_\varepsilon\, \varepsilon^{1/4}, \end{aligned}$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$. It then follows

$$\|\tilde{u}_\varepsilon - w_\varepsilon\|_{C([0,\,T];\,H)} \leq C\, \mathcal{M}_\varepsilon\, \varepsilon^{1/4}, \ \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.8}$$

Let us denote by $R(t,\varepsilon) = \tilde{v}(t) - w_\varepsilon(t)$ which clearly is the strong solution of the problem

$$\begin{cases} R'(t,\varepsilon) + A_0 R(t,\varepsilon) = \varepsilon\, A_1 w_\varepsilon(t) + \mathcal{F}(t,\varepsilon), \quad t > 0, \\ R(0,\varepsilon) = R_0, \end{cases} \tag{6.9}$$

where $R_0 = u_0 - w_0$ and

$$\mathcal{F}(t,\varepsilon) = \tilde{f}(t) - \int_0^\infty K(t,\tau,\varepsilon) \tilde{f}_\varepsilon(\tau)\, d\tau - f_0(t,\varepsilon)\, u_{1\,\varepsilon}. \tag{6.10}$$

Taking the inner product by R in the equation in (6.9) and then integrating, we obtain

$$|R(t,\varepsilon)|^2 + 2\int_0^t \left| A_0^{1/2} R(s,\varepsilon) \right|^2 ds$$

$$= |R_0|^2 + 2\int_0^t |\mathcal{F}(s,\varepsilon)|\, |R(s,\varepsilon)|\, ds + 2\,\varepsilon \int_0^t (A_1 w_\varepsilon(s), R(s,\varepsilon))\, ds,$$

for all $t \geq 0$. Using (6.4), from the last equality, we get

$$|R(t,\varepsilon)|^2 + \int_0^t \left|A_0^{1/2} R(s,\varepsilon)\right|^2 ds \leq |R_0|^2$$

$$+2\int_0^t |\mathcal{F}(s,\varepsilon)|\, |R(s,\varepsilon)|\, ds + \gamma^2\,\varepsilon^2 \int_0^t \left|A_0^{1/2} w_\varepsilon(s)\right|^2 ds, \tag{6.11}$$

for all $t \geq 0$. Applying Lemma 1 to (6.11), we obtain

$$|R(t,\varepsilon)| + \left(\int_0^t \left|A_0^{1/2} R(s,\varepsilon)\right|^2 ds\right)^{1/2} \leq |R_0|$$

$$+\int_0^t |\mathcal{F}(s,\varepsilon)|\, ds + \gamma\,\varepsilon \left(\int_0^t \left|A_0^{1/2} w_\varepsilon(s)\right|^2 ds\right)^{1/2}, \ \forall t \geq 0. \tag{6.12}$$

From (6.6), we deduce

$$|R_0| \leq |u_{0\,\varepsilon} - u_0| + \int_0^\infty e^{-s}\, |\tilde{u}_\varepsilon(2\varepsilon s) - u_{0\,\varepsilon}|\, ds \leq |u_{0\,\varepsilon} - u_0|$$

$$+\int_0^\infty e^{-s} \int_0^{2\varepsilon s} \left|\tilde{u}_\varepsilon'(\tau)\right| d\tau\, ds \leq |u_{0\,\varepsilon} - u_0| + C\,\mathcal{M}_\varepsilon\,\varepsilon^{1/2}, \tag{6.13}$$

for all $\varepsilon \in (0, \varepsilon_0]$. Using (x) in Lemma 4 and (6.5), we get

$$\left|\tilde{f}(t) - \int_0^\infty K(t,\tau,\varepsilon)\, \tilde{f}_\varepsilon(\tau)\, d\tau\right|$$

$$\leq \left|\tilde{f}(t) - \tilde{f}_\varepsilon(t)\right| + \int_0^\infty K(t,\tau,\varepsilon) \left|\tilde{f}_\varepsilon(t) - \tilde{f}_\varepsilon(\tau)\right| d\tau \leq \left|\tilde{f}(t) - \tilde{f}_\varepsilon(t)\right|$$

$$+C(T,p)\, \|f_\varepsilon'\|_{L^p(0,T;H)}\, \varepsilon^{(p-1)/2p}, \ \forall t \geq 0, \ \forall \varepsilon \in (0, \varepsilon_0]. \tag{6.14}$$

As $e^\tau \lambda(\sqrt{\tau}) \leq C$ for all $\tau \geq 0$, we have

$$\int_0^t \exp\left\{\frac{3\tau}{4\varepsilon}\right\} \lambda\left(\sqrt{\frac{\tau}{\varepsilon}}\right) d\tau \leq C\,\varepsilon \int_0^{\frac{t}{\varepsilon}} e^{-\tau/4}\, d\tau \leq C\,\varepsilon \int_0^\infty e^{-\tau/4}\, d\tau \leq C\varepsilon$$

and
$$\int_0^t \lambda\left(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\right)\,d\tau \leq \varepsilon \int_0^\infty \lambda\left(\frac{1}{2}\sqrt{\tau}\right)\,d\tau \leq C\,\varepsilon,$$

for all $t \geq 0$. Hence
$$\left|\int_0^t f_0(\tau,\varepsilon)\,d\tau\, u_{1\,\varepsilon}\right| \leq C\,\varepsilon\,|u_{1\,\varepsilon}|, \quad \forall t \geq 0. \tag{6.15}$$

Using (6.10), from (6.14) and (6.15), we get
$$\int_0^t |\mathcal{F}(s,\varepsilon)|\; ds \leq C\,\left(\mathcal{M}_\varepsilon\,\varepsilon^{(p-1)/2p} + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.16}$$

for every $t \in [0,T]$ and every $\varepsilon \in (0,\varepsilon_0]$.

As $A_0^{1/2}$ is closed, using (6.6), we obtain
$$\left|A_0^{1/2} w_\varepsilon(t)\right| \leq \int_0^\infty K(t,\tau,\varepsilon)\,\left|A_0^{1/2}\tilde{u}_\varepsilon(\tau)\right|\,d\tau \leq C\,\mathcal{M}_\varepsilon, \tag{6.17}$$

for every $t \in [0,T]$ and every $\varepsilon \in (0,\varepsilon_0]$.

Thanks to (6.13), (6.16) and (6.17), from (6.12), it follows that
$$\|R\|_{C([0,T];\,H)} + \left\|A_0^{1/2} R\right\|_{L^2(0,\,T;\,H)}$$
$$\leq C\,\left(\mathcal{M}_\varepsilon\,\varepsilon^{(p-1)/2\,p} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.18}$$

for every $\varepsilon \in (0,\varepsilon_0]$. Finally, from (6.8) and (6.18), it follows that
$$\|\tilde{u}_\varepsilon - \tilde{v}\|_{C([0,T];H)} \leq \|\tilde{u}_\varepsilon - w_\varepsilon\|_{C([0,T];H)} + \|R\|_{C([0,T];H)}$$
$$\leq C\,\left(\mathcal{M}_\varepsilon\,\varepsilon^\beta + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;H)}\right), \tag{6.19}$$

for every $\varepsilon \in (0,\varepsilon_0]$. According to Theorems 9 and 10, we have that $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$ and $\tilde{v}(t) = v(t)$ for $t \in [0,T]$. Therefore, from (6.19), we deduce (6.1).

Proof of (6.2). If $u_{1\,\varepsilon} \in D\left(A_0^{1/2}\right)$, then, using (vi), $(viii)$, (x) in Lemma 4 and (6.7), we get
$$\|\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\|_H = \left\|\tilde{u}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau\right\|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\, \|\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\|_H \, d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left| \int_t^\tau \|\tilde{u}'_\varepsilon(s)\|_H \, ds \right| d\tau$$

$$\leq \|\tilde{u}'_\varepsilon\|_{C([0,\infty);\, H)} \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau| \, d\tau \leq C\, \mathcal{M}_{1\,\varepsilon}\, \varepsilon^{1/2},$$

for every $t \in [0,T]$ and every $\varepsilon \in (0,\varepsilon_0]$. This yields

$$\|\tilde{u}_\varepsilon - w_\varepsilon\|_{C([0,T];\, H)} \leq C\, \mathcal{M}_{1\,\varepsilon}\, \varepsilon^{1/2}, \quad \forall \varepsilon \in (0,\varepsilon_0].$$

As, for $p \in (1,\infty]$, we have $(p-1)/2p \leq 1/2$, the proof of (6.2) follows in the same way as the proof of (6.1).

Proof of (6.3). Using (vi), $(viii)$, (x) in Lemma 4 and (6.7), we get

$$\left| A_0^{1/2}(\tilde{u}_\varepsilon(t) - w_\varepsilon(t) \right| \leq \int_0^\infty K(t,\tau,\varepsilon) \left| A_0^{1/2}(\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau) \right| d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left| \int_\tau^t \left\| A_0^{1/2} \tilde{u}'_\varepsilon(s) \right\|_H \, ds \right| d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|^{1/2} \left| \int_\tau^t \left\| A_0^{1/2} \tilde{u}'_\varepsilon(s) \right\|_H^2 \, ds \right|^{1/2} d\tau$$

$$\leq C\, \mathcal{M}_{1\,\varepsilon}\, \varepsilon^{1/4}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon_0].$$

Hence $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$, for $t \in [0,T]$, and therefore

$$\left\| A_0^{1/2}\,(u_\varepsilon - w_\varepsilon) \right\|_{C([0,T];\, H)} \leq C\, \mathcal{M}_{1\,\varepsilon}\, \varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.20}$$

From (6.18), it follows that

$$\left\| A_0^{1/2} R \right\|_{L^2(0,T)\,;H)} \leq C \Big(\mathcal{M}_\varepsilon\, \varepsilon^{(p-1)/2\,p}$$

$$+ |u_{0\,\varepsilon} - u_0| + C\, \|f_\varepsilon - f\|_{L^p(0,T;H)} \Big), \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.21}$$

Finally, (6.20) and (6.21) imply (6.3) and this completes the proof. □

Remark 6.1. If, in the conditions of Theorem 12, we assume that $f, f_\varepsilon \in W^{1,\infty}(0,T,H)$, then (6.1), (6.2) and (6.3) take the form

$$\|u_\varepsilon - v\|_{C([0,T];H)} \leq C\,\left(\mathcal{M}_\varepsilon\,\varepsilon^{1/4} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^\infty(0,T;H)}\right),$$

where

$$\mathcal{M}_\varepsilon = \left|A_0^{1/2} u_{0\,\varepsilon}\right| + |u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,\infty}(0,T;H)},$$

$$\|u_\varepsilon - v\|_{C([0,T];H)} \leq C\,\left(\mathcal{M}_{1\,\varepsilon}\,\varepsilon^{1/2} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^\infty(0,T;H)}\right),$$

and

$$\|A_0^{1/2} u_\varepsilon - A_0^{1/2} v\|_{L^2(0,T;\,H)}$$
$$\leq C\,\left(\mathcal{M}_{1\,\varepsilon}\,\varepsilon^{1/4} + |u_{0\,\varepsilon} - u_0| + |f_\varepsilon - f\|_{L^\infty(0,T;H)}\right),$$

with

$$\mathcal{M}_{1\,\varepsilon} = \left|A_0^{1/2} u_{1\,\varepsilon}\right| + |A_0 u_{0\,\varepsilon}| + |A_1 u_{0\,\varepsilon}| + \|f\|_{W^{1,\infty}(0,T;\,H)}.$$

for all $\varepsilon \in (0, \varepsilon_0]$.

Theorem 13. *Let $T > 0$ and $p \in (1, \infty]$. Suppose that the operators A_0 and A_1 satisfy* **(H1)** *and* **(H2)**. *If*

$$u_0, u_{0\varepsilon}, A_0 u_0, A_1 u_{0\varepsilon}, A_0 u_{0\varepsilon}, u_{1\varepsilon}, f(0), f_\varepsilon(0) \in D(A_0)$$

and

$$f, f_\varepsilon \in W^{2,p}(0,T;H),$$

then there exist $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1) \in (0,1)$ and $C = C(T, p, \omega_0, \omega_1) > 0$ such that

$$\left\|u_\varepsilon' - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}}\right\|_{C([0,\,T];\,H)} \leq C\,\left(\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{(p-1)/2p} + D_\varepsilon\right), \tag{6.22}$$

$$\left\|A_0^{1/2}\left(u_\varepsilon' - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}}\right)\right\|_{L^2(0,\,T;\,H)} \leq C\,\left(\mathcal{M}_{2\,\varepsilon}\,\varepsilon^{\beta} + D_\varepsilon\right), \tag{6.23}$$

where v and u_ε are the strong solutions of the problems (P_0) and (P_ε) respectively, $\beta = \min\{1/4, (p-1)/2p\}$, $\alpha_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon) u_{0\,\varepsilon}$,

$$D_\varepsilon = \|f_\varepsilon - f\|_{W^{1,\,p}(0,\,T;\,H)} + |A_0(u_{0\varepsilon} - u_0)|\,,$$

$$\mathcal{M}_{2\,\varepsilon} = |A(\varepsilon) u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{2,p}(0,\,T;\,H)} + |A_1 u_{0\,\varepsilon}| + |A(\varepsilon)\alpha_\varepsilon|\,.$$

Proof. Within this proof, for $\tilde{u}_\varepsilon$, $\tilde{v}$, $\tilde{f}$ and $\tilde{f}_\varepsilon$, we will use the same notations as in the proof of Theorem 12.

Let us denote by

$$\tilde{z}_\varepsilon(t) = \tilde{u}'_\varepsilon(t) + \alpha_\varepsilon e^{-\frac{t}{\varepsilon}}, \quad \alpha_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon)u_{0\,\varepsilon}.$$

If $u_{1\,\varepsilon} + \alpha_\varepsilon \in D\,(A_0)$ and $f \in W^{2,1}(0,T;H)$, then, due to (6.4) and (6.5), $u_{1\,\varepsilon} + \alpha_\varepsilon \in D\,(A(\varepsilon))$ and $\tilde{f} \in W^{2,1}(0,\infty;H)$. According to Theorem 10, $\tilde{z}_\varepsilon$ is the strong solution in H to the problem

$$\begin{cases} \varepsilon\tilde{z}''_\varepsilon(t) + \tilde{z}'_\varepsilon(t) + A(\varepsilon)\tilde{z}_\varepsilon(t) = \tilde{\mathcal{F}}(t,\varepsilon), \quad t > 0, \\ \tilde{z}_\varepsilon(0) = f_\varepsilon(0) - A(\varepsilon)u_{0\,\varepsilon}, \quad \tilde{z}'_\varepsilon(0) = 0, \end{cases}$$

where

$$\tilde{\mathcal{F}}(t,\varepsilon) = \tilde{f}'_\varepsilon(t) + e^{-t/\varepsilon}\,A(\varepsilon)\alpha_\varepsilon.$$

From Lemma 3 and (6.4), it follows that

$$\tilde{z}_\varepsilon \in W^{2,\infty}(0,\infty;H),\ A_0^{1/2}\tilde{z}_\varepsilon \in W^{1,2}(0,\infty;H),\ A(\varepsilon)\tilde{z}_\varepsilon \in L^\infty(0,\infty;H).$$

Moreover, from the same lemma, (6.4) and (6.5), we get

$$\begin{aligned} &\|A_0^{1/2}\tilde{z}_\varepsilon\|_{C([0,\,\infty];\,H)} + \|\tilde{z}'_\varepsilon\|_{C([0,\,\infty);\,H)} \\ &+ \left\|A_0^{1/2}\tilde{z}'_\varepsilon\right\|_{L^2(0,\,\infty;\,H)} \le C\,\mathcal{M}_{2\,\varepsilon}, \quad \forall\varepsilon \in (0,\varepsilon_0]. \end{aligned} \tag{6.24}$$

According to Theorem 4, the function

$$w_{1\,\varepsilon}(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)d\tau$$

is a strong solution of

$$\begin{cases} w'_{1\,\varepsilon}(t) + A(\varepsilon)w_{1\varepsilon}(t) = \mathcal{F}_1(t,\varepsilon), \quad t > 0, \\ w_{1\,\varepsilon}(0) = \int_0^\infty e^{-\tau}\,\tilde{z}_\varepsilon(2\,\varepsilon\,\tau)d\tau, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\left(\tilde{f}'_\varepsilon(\tau) + e^{-\frac{\tau}{\varepsilon}}\,A(\varepsilon)\alpha_\varepsilon\right)\,d\tau.$$

Moreover,

$$\left|A_0^{1/2}w_{1\,\varepsilon}(t)\right| \le \int_0^\infty K(t,\tau,\varepsilon)\,\left|A_0^{1/2}\tilde{z}_\varepsilon(\tau)\right|\,d\tau \le C\mathcal{M}_{2\,\varepsilon}, \tag{6.25}$$

for all $t \geq 0$. Using (vi), $(viii)$, (x) in Lemma 4 and (6.24), we get

$$\|\tilde{z}_\varepsilon(t) - w_{1\,\varepsilon}(t)|_H = \left\| \tilde{z}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\, \tilde{z}_\varepsilon(\tau)\, d\tau \right\|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\, \|\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau)\|_H \, d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left| \int_t^\tau \|\tilde{z}'_\varepsilon(s)\|_H \, ds \right| d\tau$$

$$\leq \|\tilde{z}'_\varepsilon\|_{C([0,\infty);\,H)} \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|\, d\tau \leq C\, \mathcal{M}_{2\,\varepsilon}\, \varepsilon^{1/2},$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$,

$$\left\| A_0^{1/2} \left(\tilde{z}_\varepsilon(t) - w_{1\,\varepsilon}(t) \right) \right\|_H$$

$$= \left\| A_0^{1/2} \tilde{z}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\, A_0^{1/2} \tilde{z}_\varepsilon(\tau)\, d\tau \right\|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left\| A_0^{1/2} \left(\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau) \right) \right\|_H d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon) \left| \int_t^\tau \|A_0^{1/2} \tilde{z}'_\varepsilon(s)\|_H \, ds \right| d\tau$$

$$\leq \|A_0^{1/2} \tilde{z}'_\varepsilon\|_{L^2(0,\infty;\,H)} \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|^{1/2}\, d\tau \leq C\, \mathcal{M}_{2\,\varepsilon}\, \varepsilon^{1/4},$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\varepsilon_0]$. It then follows that

$$\|\tilde{z}_\varepsilon - w_{1\,\varepsilon}\|_{C([0,T];\,H)} \leq C\, \mathcal{M}_{2\,\varepsilon}\, \varepsilon^{1/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \tag{6.26}$$

$$\left\| A_0^{1/2} \left(\tilde{z}_\varepsilon - w_{1\,\varepsilon} \right) \right\|_{L^2(0,T;\,H)} \leq C\, \mathcal{M}_{2\,\varepsilon}\, \varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.27}$$

Let $R_1(t,\varepsilon) = \tilde{v}'(t) - w_{1\varepsilon}(t)$. If $f(0) - A_0 u_0 \in D(A_0)$ and $f \in W^{2,1}(0,T;H)$, then, according to Theorem 3.1, $\tilde{v} \in W^{2,\infty}(0,\infty;H)$, $A_0^{1/2}\tilde{v} \in W^{1,2}(0,\infty;H)$. Therefore $R_1 \in W^{1,\infty}(0,\infty;H)$ and

$$\begin{cases} R'_1(t,\varepsilon) + A_0 R_1(t,\varepsilon) = \tilde{f}'(t) - \mathcal{F}_1(t,\varepsilon) + \varepsilon A_1 w_{1\,\varepsilon}(t), \quad t > 0, \\ R_1(0,\varepsilon) = f(0) - A_0 u_0 - w_{1\,\varepsilon}(0). \end{cases}$$

Similarly to (6.12), we deduce

$$|R_1(t,\varepsilon)| + \left(\int_0^t \left|A_0^{1/2} R_1(s,\varepsilon)\right|^2 ds\right)^{1/2} \le |R_1(0,\varepsilon)|$$

$$+ \int_0^t \left|\tilde{f}'(s) - \mathcal{F}_1(s,\varepsilon)\right| ds + \gamma\,\varepsilon \left(\int_0^t \left|A_0^{1/2} w_{1\,\varepsilon}(s)\right|^2 ds\right)^{1/2}, \tag{6.28}$$

for all $t \ge 0$. Using (6.24), we get:

$$|R_1(0,\varepsilon)| \le |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\,\varepsilon})| + \varepsilon\, |A_1 u_{0\,\varepsilon}|$$

$$+ \int_0^\infty e^{-s}\, |\tilde{z}_\varepsilon(2\,\varepsilon\, s) - \tilde{z}_\varepsilon(0)|\, ds$$

$$\le C\, D_\varepsilon + \varepsilon\, |A_1 u_{0\,\varepsilon}| + \mathcal{M}_{2\,\varepsilon}\,\varepsilon \le C\, D_\varepsilon + \mathcal{M}_{2\,\varepsilon}\,\varepsilon, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.29}$$

As

$$\left|\tilde{f}'(s) - \mathcal{F}_1(s,\varepsilon)\right| \le \left|\tilde{f}'(s) - \tilde{f}_\varepsilon'(s)\right| + \int_0^\infty K(s,\tau,\varepsilon)\, \left|\tilde{f}_\varepsilon'(\tau) - \tilde{f}_\varepsilon'(s)\right| d\tau$$

$$+ \int_0^\infty K(s,\tau,\varepsilon)\, e^{-\frac{\tau}{\varepsilon}}\, d\tau\, \left| A(\varepsilon)\alpha_\varepsilon\right|,$$

then, due to (ix), (xi) in Lemma 4, we obtain

$$\int_0^t \left|\tilde{f}'(s) - \mathcal{F}_1(s,\varepsilon)\right| ds \le C\left(D_\varepsilon + \mathcal{M}_{2\,\varepsilon}\,\varepsilon^{(p-1)/2p} + \left| A(\varepsilon)\alpha_\varepsilon\right|\,\varepsilon\right)$$

$$\le C\left(D_\varepsilon + \mathcal{M}_{2\,\varepsilon}\,\varepsilon^{(p-1)/2p}\right), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{6.30}$$

Using (6.25), (6.29), (6.30), from (6.28) we get

$$\|R_1\|_{C([0,T];\,H)} + \left\|A_0^{1/2} R_1\right\|_{L^2(0,T;\,H)} \le C\left(D_\varepsilon + \mathcal{M}_{2\,\varepsilon}\,\varepsilon^{(p-1)/2p}\right), \tag{6.31}$$

for all $\varepsilon \in (0,1]$.

Finally, as (6.26), (6.27) and (6.31) imply (6.22) and (6.23), the proof is complete. $\square$

Remark 6.2. If, in the conditions of Theorem 13, we assume that $f, f_\varepsilon \in W^{2,\infty}(0,T,H)$, then (6.22) and (6.23) take the form

$$\left\| u_\varepsilon' - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}} \right\|_{C([0,T];\,H)} \leq C\, \left(\mathcal{M}_{2\,\varepsilon}\, \varepsilon^{1/2} + D_\varepsilon \right),$$

$$\left\| A_0^{1/2} \left(u_\varepsilon' - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}} \right) \right|_{L^2(0,T;\,H)} \leq C\, \left(\mathcal{M}_{2\,\varepsilon}\, \varepsilon^{1/4} + D_\varepsilon \right),$$

$$D_\varepsilon = \| f_\varepsilon - f|_{W^{1,\infty}(0,T;\,H)} + |A_0(u_{0\varepsilon} - u_0)|\,,$$

$$\mathcal{M}_{2\,\varepsilon} = |A(\varepsilon) u_{1\,\varepsilon}| + \| f_\varepsilon|_{W^{2,\infty}(0,T;\,H)} + |A_1 u_{0\,\varepsilon}| + |A(\varepsilon)\alpha_\varepsilon|\,.$$

7 An Example

Let $\Omega \subset R^n$ be an open bounded set with C^1 boundary $\partial\Omega$. In the real Hilbert space $L^2(\Omega)$, with the usual inner product

$$(u,v) = \int_\Omega u(x)\, v(x)\, dx,$$

we consider the following Cauchy problem

$$\begin{cases} \varepsilon \partial_t^2\, u_\varepsilon(x,t) + \partial_t\, u_\varepsilon(x,t) + A_0 u_\varepsilon(x,t) + \varepsilon A_1 u_\varepsilon(x,t) = f(x,t), \\ x \in \Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad \partial_t\, u_\varepsilon(x,0) = u_{1\varepsilon}(x) \end{cases} \tag{7.1}$$

where $D(A_i) = H^2(\Omega) \cap H_0^1(\Omega)$, $i = 0,1$,

$$A_0 u(x) = -\sum_{i,j=1}^n \partial_{x_i} \left(a_{ij}(x) \partial_{x_j} u(x) \right) + a(x) u(x), \quad u \in D(A_0),$$

$$a_{ij} \in C^1(\overline{\Omega}),\ a \in C(\overline{\Omega}),\ a(x) \geq 0,\ a_{ij}(x) = a_{ji}(x), \quad x \in \overline{\Omega}, \tag{7.2}$$

and

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i\, \xi_j \geq a_0\, |\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad a_0 > 0. \tag{7.3}$$

$$A_1 u(x) = -\sum_{i,j=1}^n \partial_{x_i} \left(b_{ij}(x) \partial_{x_j} u(x) \right) + b(x) u(x) + \int_\Omega K(x,y) u(y) dy,$$

for $u \in D(A_1)$,

$$K : \Omega \times \Omega \mapsto R, \quad K \in L^2(\Omega \times \Omega), \tag{7.4}$$

$$b_{ij} \in C^1(\overline{\Omega}),\ b \in C(\overline{\Omega}),\ b_{ij}(x) = b_{ji}(x), \quad x \in \overline{\Omega}, \tag{7.5}$$

$$|b(x)| \le b_1 a(x), \quad \left|\sum_{i,j=1}^{n} b_{ij}(x)\xi_i\,\xi_j\right| \le b_0 \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\,\xi_j \tag{7.6}$$

for $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$. Under the hypotheses (7.2)-(7.3), the operator A_0 is positive and self-adjoint with $D(A_0^{1/2}) = H_0^1(\Omega)$ and

$$\|A_0^{1/2}u\|^2_{L^2(\Omega)} = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{x_i}u(x)\,\partial_{x_j}u(x) + a(x)u^2(x)\right)dx,$$

for $u \in H_0^1(\Omega)$. If (7.5) holds, the operator A_1 is self-adjoint with

$$\|A_1^{1/2}u\|^2_{L^2(\Omega)} = \int_\Omega \left(\sum_{i,j=1}^{n} b_{ij}(x)\,\partial_{x_i}u(x)\,\partial_{x_j}u(x) + b(x)u^2(x)\right)dx$$

$$+ \int_\Omega\int_\Omega K(x,y)u(x)u(y)dy\,dx, \quad \forall u \in H_0^1(\Omega).$$

Moreover, (7.2)-(7.6) imply (**H2**) with

$$\omega_1 = \max\{b_0, b_1\} + \|K\|_{L_2(\Omega\times\Omega)}/\omega_0.$$

Let us now consider the unperturbed problem associated to (7.1)

$$\begin{cases} \partial_t\, v(x,t) + A_0 v(x,t) = f(x,t), & x \in \Omega,\ t > 0, \\ v(x,0) = u_0(x). \end{cases} \tag{7.7}$$

Using Theorem 12, we obtain:

Theorem 14. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^1 boundary $\partial\Omega$. Let $T > 0$ and $p \in (1,\infty]$. Let us assume that* (7.2)-(7.6) *are satisfied. If*

$$u_0, u_{0\,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega), \quad u_{1\,\varepsilon} \in L^2(\Omega), \quad f, f_\varepsilon \in W^{1,p}(0,T; L^2(\Omega)),$$

then there exist $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1) \in (0,1)$ and $C = C(T,p,n,\omega_0,\omega_1) > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, we have

$$\|u_\varepsilon - v\|_{C([0,T];L^2(\Omega))}$$

$$\leq C\,\left(\widetilde{\mathcal{M}}_\varepsilon\,\varepsilon^\beta + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;L^2(\Omega))}\right),$$

where u_ε *and* v *are the strong solutions of* (7.1) *and* (7.7) *respectively,*

$$\beta = \min\{1/4, (p-1)/2p\}$$

and

$$\widetilde{\mathcal{M}}_\varepsilon = \left|A_0^{1/2} u_{0\,\varepsilon}\right| + |u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,p}(0,T;L^2(\Omega))}.$$

If, in addition, $u_{1\,\varepsilon} \in H_0^1(\Omega)$*, then*

$$\|u_\varepsilon - v\|_{C([0,T];L^2(\Omega))}$$

$$\leq C\,\left(\widetilde{\mathcal{M}}_{1\,\varepsilon}\,\varepsilon^{(p-1)/2p} + |u_{0\,\varepsilon} - u_0| + \|f_\varepsilon - f\|_{L^p(0,T;L^2(\Omega))}\right),$$

for each $\varepsilon \in (0, \varepsilon_0]$*, where* $\beta = \min\{1/4, (p-1)/2p\}$ *and*

$$\widetilde{\mathcal{M}}_{1\,\varepsilon} = \left|A_0^{1/2} u_{1\,\varepsilon}\right| + |A_0 u_{0\,\varepsilon}| + |A_1 u_{0\,\varepsilon}| + \|f_\varepsilon\|_{W^{1,p}(0,\,T;\,L^2(\Omega))}.$$

Using Theorem 13, we deduce:

Theorem 15. *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded set with* C^1 *boundary* $\partial\Omega$*. Let* $T > 0$ *and* $p \in (1, \infty]$*. Let us assume that* (7.2)-(7.6) *are satisfied. If*

$$u_0, u_{0\varepsilon}, A_0 u_0, A_1 u_{0\varepsilon}, A_0 u_{0\varepsilon}, u_{1\varepsilon}, f(0), f_\varepsilon(0) \in H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$f, f_\varepsilon \in W^{2,p}(0,T;L^2(\Omega)),$$

then there exist $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1) \in (0,1)$ *and* $C = C(T, p, n, \omega_0, \omega_1) > 0$ *such that, for every* $\varepsilon \in (0, \varepsilon_0]$*, we have*

$$\left\|u_\varepsilon' - v' + \alpha_\varepsilon\, e^{-\frac{t}{\varepsilon}}\right\|_{C([0,\,T];\,L^2(\Omega))} \leq C\,\left(\widetilde{\mathcal{M}}_{2\,\varepsilon}\,\varepsilon^{(p-1)/2p} + \widetilde{D}_\varepsilon\right),$$

where v *and* u_ε *are the strong solutions of* (7.1) *and* (7.7) *respectively,* $\beta = \min\{1/4, (p-1)/2p\}$*,* $\alpha_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon) u_{0\,\varepsilon}$*,*

$$\widetilde{D}_\varepsilon = \|f_\varepsilon - f\|_{W^{1,\,p}(0,\,T;\,H_0^1(\Omega))} + |A_0(u_{0\varepsilon} - u_0)|\,,$$

$$\widetilde{\mathcal{M}}_{2\,\varepsilon} = |A(\varepsilon) u_{1\,\varepsilon}| + \|f_\varepsilon\|_{W^{2,p}(0,\,T;\,H_0^1(\Omega))} + |A_1 u_{0\,\varepsilon}| + |A(\varepsilon)\alpha_\varepsilon|\,.$$

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A VIABILITY RESULT FOR EVOLUTION EQUATIONS ON LOCALLY CLOSED GRAPHS*

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Abstract

Using a tangency condition expressed with a set of integrals, we establish several necessary and sufficient conditions for viability referring to evolution equations on locally closed graphs.

keywords: Differential inclusion, locally closed graph, tangent set, tangency condition, multi-valued mapping, viability.

1 Introduction

Let X be a real Banach space, let $I \subseteq \mathrm{R}$ be a nonempty and bounded interval and let $K : I \rightsquigarrow X$ and $F : \mathcal{K} \rightsquigarrow X$ be two multi-functions with nonempty values, where $\mathcal{K} := \mathrm{graph}(K)$. Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup $\{S(t);\ t \geq 0\}$.

Our aim here is to prove some new necessary and sufficient conditions in order that $\mathcal{K}$ be viable with respect to $A + F$. This paper is an extension of the results established by Necula-Popescu-Vrabie [7].

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To be more precise, let us consider the Cauchy Problem

$$\begin{cases} u'(t) \in Au(t) + F(t, u(t)) \\ u(\tau) = \xi. \end{cases} \tag{1.1}$$

Definition 1.1. By a *mild solution* of (1.1) on $[\tau, T] \subseteq I$, we mean a function $u \in C([\tau, T]; X)$ satisfying $(t, u(t)) \in \mathcal{K}$, $u(\tau) = \xi$ and for which there exists $f \in L^1(\tau, T; X)$ with $f(t) \in F(t, u(t))$ a.e. for $t \in [\tau, T]$ and

$$u(t) = S(t-\tau)\xi + \int_\tau^t S(t-s)f(s)\,ds \tag{1.2}$$

for each $t \in [\tau, T]$.

Definition 1.2. We say that the graph, $\mathcal{K}$, of $K : I \rightsquigarrow X$, is *mild viable* with respect to $A + F$, where $F : \mathcal{K} \rightsquigarrow X$, if for each $(\tau, \xi) \in \mathcal{K}$, there exists $T > \tau$, such that $[\tau, T] \subseteq I$ and (1.1) has at least one mild solution $u : [\tau, T] \to X$. If $T \in (\tau, \sup I)$ can be taken arbitrary, we say that $\mathcal{K}$ is *globally mild viable* with respect to $A + F$.

The first two sections of the paper are concerned with some prerequisites and basic concepts and results needed in the sequel. In Section 3 we prove the main necessary condition of viability, in Section 4 we give a relationship between two tangency conditions, Section 5 contains the statement of the two sufficient conditions for viability and the statement and proof of a technical approximation lemma, while in Section 6, we give the proofs of Theorems 5.1 and 5.2.

2 Preliminaries

If (Y, d) is a metric space, $y \in Y$ and $r > 0$, $D(y, r)$ denotes the closed ball with center y and radius $r > 0$, i.e. $D(y, r) = \{x \in Y;\ d(y, x) \le r\}$, while $S(y, r)$ denotes the open ball with center y and radius $r > 0$, i.e. $S(y, r) = \{x \in Y;\ d(y, x) < r\}$. If $B \subseteq Y$ and $C \subseteq Y$, we denote by

$$\operatorname{dist}(y, C) := \inf\{d(y, z);\ z \in C\}$$

and by

$$\operatorname{dist}(B, C) := \inf\{d(x, y);\ x \in B,\ y \in C\}.$$

Also $\mathcal{B}(Y)$ denotes the family of all bounded subsets of Y.

Definition 2.1. Let $Y \subseteq X$ be nonempty. The function $\beta_Y : \mathcal{B}(X) \to \mathbb{R}_+$, defined by

$$\beta_Y(B) := \inf \left\{ \varepsilon > 0; \exists x_1, x_2, \ldots, x_{n(\varepsilon)} \in Y,\ B \subseteq \bigcup_{i=1}^{n(\varepsilon)} D(x_i, \varepsilon) \right\},$$

is called the *Hausdorff measure of noncompactness on X subordinated to Y*. If $Y = X$, we simply denote β_X by β, and we simply call it the *Hausdorff measure of noncompactness on X.*

Remark 2.1. We have the following properties:

(i) for each $B \in \mathcal{B}(X)$ and $r > 0$ with $B \subseteq D(0, r)$, we have $\beta(B) \le r$;

(ii) $\beta(B) = 0$ if and only if B is relatively compact;

(iii) the restriction of β_Y to $\mathcal{B}(Y)$ coincides with the Hausdorff measure of noncompactness on Y;

(iv) for each $B \in \mathcal{B}(Y)$ we have $\beta(B) \le \beta_Y(B) \le 2\,\beta(B)$.

The next lemma is due to Mönch [4].

Lemma 2.1. *Let X be a separable Banach space and $\{f_m;\ m \in \mathbb{N}\}$ a subset in $L^1(\tau, T; X)$ for which there exists $\ell \in L^1(\tau, T; \mathbb{R}_+)$ such that*

$$\|f_m(s)\| \le \ell(s)$$

for each $m \in \mathbb{N}$ and a.e. for $s \in [\,\tau, T\,]$. Then the mapping

$$s \mapsto \beta(\{f_m(s);\ m \in \mathbb{N}\})$$

is integrable on $[\,\tau, T\,]$ and, for each $t \in [\,\tau, T\,]$, we have

$$\beta\left(\left\{\int_\tau^t f_m(s)\,ds;\ m \in \mathbb{N}\right\}\right) \le \int_\tau^t \beta(\{f_m(s);\ m \in \mathbb{N}\})\,ds. \tag{2.1}$$

For further details on the Haussdorf measure of noncompactness see Cârjă, Necula, Vrabie [3], Section 2.7, pp. 48∼53.

Let X be a real Banach space, $I \subseteq \mathrm{R}$ a nonempty and bounded interval, $K : I \rightsquigarrow X$ a multi-function with nonempty values and let $\mathcal{K} := \mathrm{graph}(K)$. Here and thereafter, $\mathcal{K}$ is conceived as a metric space, whose metric, d, is defined by $d((\tau,\xi),(\theta,\mu)) = \max\{|\tau-\theta|, \|\xi-\mu\|\}$, for all $(\tau,\xi),(\theta,\mu) \in \mathcal{K}$. Also, λ denotes the Lebesgue measure on $\mathbb{R}$. Furthermore, whenever we will use the term *strongly-weakly* we will mean that the domain of the multi-function in question is equipped with the strong topology, while the range is equipped with the weak topology. Otherwise, both domain and range are endowed with the strong, i.e. norm, topology.

Definition 2.2. The multi-function $F : \mathcal{K} \rightsquigarrow X$ is called (*strongly-weakly*) *almost u.s.c.* if for each $\varepsilon > 0$ there exists an open set $\mathcal{O}_\varepsilon \subseteq I$ such that $\lambda(\mathcal{O}_\varepsilon) \leq \varepsilon$ and $F_{|[(I\setminus\mathcal{O}_\varepsilon)\times X]\cap\mathcal{K}}$ is (strongly-weakly) u.s.c.

Definition 2.3. The multi-function $F : \mathcal{K} \rightsquigarrow X$ is called *integrally-bounded* if for each $(\tau,\xi) \in \mathcal{K}$ there exist $\rho > 0$, $\delta > 0$, $\ell_1 \in L^1(I;\mathbb{R})$ and a negligible set $N_1 \subseteq I$ satisfying: for each $(t,u) \in (([\tau-\delta,\tau+\delta]\setminus N_1)\times S(\xi,\rho))\cap\mathcal{K}$, we have

$$\|F(t,u)\| \leq \ell_1(t).$$

Remark 2.2. (i) If X is separable we can choose N_1 in Definition 2.3 the same for all $(\tau,\xi) \in \mathcal{K}$ and in this case for each $(\tau,\xi) \in ((I\setminus N_1)\times X)\cap\mathcal{K}$, $F(\tau,\xi)$ is bounded.

(ii) Moreover, if, in addition, F is closed valued and almost u.s.c., then, for each continuous function $u : I \to X$ with $(t,u(t)) \in \mathcal{K}$ for each $t \in I$, the multi-function $t \mapsto F(t,u(t))$ has at least one locally integrable selection on I. The same conclusion holds true if F is closed valued, strongly-weakly almost u.s.c. and has separable range. The latter assertion follows from Pettis' Measurability Theorem 1.1.3, p. 3, in Vrabie [10].

The next special class of graphs was considered for the first time by Necula [5].

Definition 2.4. Let $K : I \rightsquigarrow X$ be a multi-function with graph, $\mathcal{K}$. By *a simple solution issuing from* $(\tau,\xi) \in \mathcal{K}$ we mean a pair of functions $(g,v) \in L^1(\tau,T;X)\times C([\tau,T];X)$ such that for all $t \in [\tau,T]$ we have $(t,v(t)) \in \mathcal{K}$ and

$$v(t) = S(t-\tau)\xi + \int_\tau^t S(t-s)g(s)\,ds$$

Definition 2.5. The graph, $\mathcal{K}$, of K is said to be *A-mild viable by itself* if for each $(\tau,\xi) \in \mathcal{K}$, there exist $T > \tau$, $\rho > 0$ and $\ell_2 \in L^1(I;\mathbb{R})$, so that for each $(\widetilde{\tau},\widetilde{\xi}) \in ([\tau,T) \times S(\xi,\rho)) \cap \mathcal{K}$, there exist a simple solution $(\widetilde{g},\widetilde{v})$ issuing from $(\widetilde{\tau},\widetilde{\xi})$ defined on $[\widetilde{\tau},\widetilde{T}]$ such that

$$\|\widetilde{g}(s)\| \leq \ell_2(s) \text{ a.e. for } s \in [\widetilde{\tau},\widetilde{T}]$$

Remark 2.3. In other words, the graph, $\mathcal{K}$, of $K : I \rightsquigarrow X$ is A-mild viable by itself if and only if, for each $(\tau,\xi) \in \mathcal{K}$, there exist $T > \tau$, $\rho > 0$ and $\ell_2 \in L^1(I;\mathbb{R})$, so that $([\tau,T) \times S(\xi,\rho)) \cap \mathcal{K}$ is mild viable with respect to $A + G$, where the multi-function $G : ([\tau,T) \times X) \cap \mathcal{K} \rightsquigarrow X$ is defined by

$$G(t,\xi) := \{v \in X;\ \|v\| \leq \ell_2(t)\},$$

for each $(t,\xi) \in ([\tau,T) \times X) \cap \mathcal{K}$

Remark 2.4. (i) Clearly, if $K : I \rightsquigarrow X$ is constant and $S(t)K \subseteq K$ for each $t \geq 0$, then $\mathcal{K}$ is A-mild viable by itself. Indeed, in this case, $\ell_2 \equiv 0$ and $G(t,\xi) \equiv \{0\}$ satisfy all the requirements in Definition 2.5.

(ii) If $\mathcal{K}$ is A-mild viable with respect to some integrally-bounded multi-function $F : \mathcal{K} \rightsquigarrow X$ then, one may easily check out that, for each $(\tau,\xi) \in \mathcal{K}$, the function G, defined as in Remark 2.3, with $\rho > 0$ given by Definition 2.3, and $\ell_2 = \ell_1$, where ℓ_1 are given by Definition 2.3, satisfies the conditions in Remark 2.2, and thus $\mathcal{K}$ is viable by itself.

Let $(\tau,\xi) \in \mathcal{K}$ and let $E \in \mathcal{B}(X)$.

Definition 2.6. We say that E is *A-right-quasi-tangent* to $\mathcal{K}$ at $(\tau,\xi) \in \mathcal{K}$ if

$$\liminf_{h\downarrow 0} \frac{1}{h}\text{dist}\left(S(h)\xi + \int_0^h S(h-s)\mathcal{F}_E\,ds, K(\tau+h)\right) = 0, \tag{2.2}$$

where

$$\mathcal{F}_E = \left\{f \in L^1_{\text{loc}}(\mathbb{R};X);\ f(s) \in E \text{ a.e. for } s \in \mathbb{R}\right\}.$$

Throughout, we denote by $\mathcal{QTS}^A_{\mathcal{K}}(\tau,\xi)$ the set of all A-right-quasi-tangent sets to $\mathcal{K}$ at (τ,ξ). If K is constant, E is A-right-quasi-tangent to $\mathcal{K}$ at (τ,ξ) if and only if it is A-quasi-tangent to K at ξ in the sense of Cârjă, Necula, Vrabie [2], [3]. The set $\mathcal{QTS}^A_{\mathcal{K}}(\tau,\xi)$ is used in Necula, Popescu, Vrabie [7] to establish necessary and sufficient conditions for viability. Next we introduce a new tangency condition which shall be used in the sequel, similar to the one used in Popescu [8].

Let $\mathcal{K}$ be A-mild viable by itself, $F : \mathcal{K} \rightsquigarrow X$ be integrally bounded and let $(\tau,\xi) \in \mathcal{K}$. Let $\ell \in L^1(I,\mathbb{R})$ such that $\ell(s) \geq \max\{\ell_1(s),\ell_2(s)\}$ a.e. for

$s \in I$ where ℓ_1 is the function from Definition 2.3 and ℓ_2 is the function from Definition 2.5.

Let us denote by $\mathcal{C}_{\tau,\xi,\ell,h}$ the set of all continuous functions $v : [\tau, \tau+h] \to X$ for which there exits $g \in L^1(\tau, \tau+h; X)$ such that (g, v) is a simple solution issuing from (τ, ξ) and $\|g(s)\| \leq \ell(s)$ a.e. for $s \in [\tau, \tau+h]$. Obviously $\mathcal{C}_{\tau,\xi,\ell,h}$ is nonempty for h small enough.

Next, let us define by $\mathcal{E}_{\tau,\xi,\ell,h}$ the set of all functions $f \in L^1(\tau, \tau+h; X)$ for which there exits $v \in \mathcal{C}_{\tau,\xi,\ell,h}$ such that $f(s) \in F(s, v(s))$ for all $s \in [\tau, \tau+h]$. If F satisfies the conditions in Remark 2.2 then $\mathcal{E}_{\tau,\xi,\ell,h}$ is nonempty for h small enough.

We consider the generalized tangency condition

$$\liminf_{h \downarrow 0} \frac{1}{h} \operatorname{dist}\left(S(h)\xi + \int_\tau^{\tau+h} S(\tau+h-s)\mathcal{E}_{\tau,\xi,\ell,h}\, ds, K(\tau+h) \right) = 0 \quad (2.3)$$

At this point, let us observe that (2.3) makes sense whenever $\mathcal{E}_{\tau,\xi,\ell,h}$ is nonempty. As we already pointed out, in order for the above set to be nonempty it is sufficient that $\mathcal{K}$ be viable by itself and $F : \mathcal{K} \rightsquigarrow X$ be integrally bounded, closed valued and almost u.s.c. Here and thereafter, when we say that (2.3) takes place, we understand that $\mathcal{K}$ is viable by itself, F is integrally bounded and $\mathcal{E}_{\tau,\xi,\ell,h} \neq \emptyset$ for h small enough (sufficiently for a certain h). The fact that (2.3) can take place even in the absence of continuity or measurability conditions for F is illustrated by the first very simple necessary condition for viability in the next section.

3 Necessary conditions for viability

The hypotheses we will use in the sequel are listed below.

(H_1) $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{S(t);\ t \geq 0\}$ of type (M, ω), i.e., $\|S(t)\| \leq Me^{\omega t}$ for each $t \geq 0$;

(H_2) the graph $\mathcal{K}$ is A-mild viable by itself;

(H_3) F has nonempty and closed values and is integrally bounded;

(H_4) $F : \mathcal{K} \rightsquigarrow X$ is almost u.s.c.;

(H_5) $F : \mathcal{K} \rightsquigarrow X$ is strongly-weakly almost u.s.c.;

(H_6) there exists a set $N \subseteq I$, with $\lambda(N) = 0$, and such that for each $(\tau, \xi) \in ((I \setminus N) \times X) \cap \mathcal{K}$, we have $F(\tau, \xi) \in \mathfrak{QTS}^A_{\mathcal{K}}(\tau, \xi)$.

(H_7) there exists a set $N \subseteq I$, with $\lambda(N) = 0$, and such that for each $(\tau, \xi) \in ((I \setminus N) \times X) \cap \mathcal{K}$, we have (2.3)

(H_8) for each $(\tau, \xi) \in \mathcal{K}$, we have (2.3).

Theorem 3.1. *If $\mathcal{K}$ is mild viable with respect to $A + F$ where F is an integrally bounded multi-function, then (H_2) and (H_8) hold true.*

Proof. First let us observe that even if F is not closed valued and almost u.s.c. the sets $\mathcal{C}_{\tau,\xi,\ell,h}$ and $\mathcal{E}_{\tau,\xi,\ell,h}$ are nonempty for h small enough. Indeed, let ρ and δ from the Definition 2.3 and $u : [\tau, T] \to S(\xi, \rho)$ be any solution of (1.1) with $T < \tau+\delta$. Then there exists $f \in L^1(\tau, T; X)$ with $f(t) \in F(t, u(t))$ a.e. for $t \in [\tau, T]$ and

$$u(t) = S(t - \tau)\xi + \int_\tau^t S(t - s)f(s)\, ds$$

for all $t \in [\tau, T]$. Hence, for each $h \in (0, T-\tau]$ we have $u \in \mathcal{C}_{\tau,\xi,\ell,h}$, $f \in \mathcal{E}_{\tau,\xi,\ell,h}$ and

$$\operatorname{dist}\left(S(h)\xi + \int_\tau^{\tau+h} S(\tau + h - s)\mathcal{E}_{\tau,\xi,\ell,h}\, ds, K(\tau + h)\right)$$
$$\leq \operatorname{dist}\left(S(h)\xi + \int_\tau^{\tau+h} S(\tau + h - s)f(s)\, ds, u(\tau + h)\right) = 0$$

and this completes the proof.

Let us remark that we have proved that for h sufficiently small

$$\{S(h)\xi + \int_\tau^{\tau+h} S(\tau + h - s)\mathcal{E}_{\tau,\xi,\ell,h}\, ds\} \cap K(\tau + h) \neq \emptyset$$

$\square$

So, under more general hypotheses on F, (H_7) is necessary in order for $\mathcal{K}$ be viable with respect to F. In that follows, we shall see that, under some additional natural assumptions on F, the converse statement is also true.

4 The relationship between (H_6) and (H_7)

Definition 4.1. We say that the multi-function $F : \mathcal{K} \rightsquigarrow X$ is almost ε–δ *l.s.c.* if for each $\gamma > 0$, there exists an open set $\mathcal{O}_\gamma \subset I$, with $\lambda(\mathcal{O}_\gamma) \leq \gamma$, and such that the mapping $(t, \xi) \mapsto F(t, \xi)$ is ε–δ l.s.c. on $((I \setminus \mathcal{O}_\gamma) \times X) \cap \mathcal{K}$.

Theorem 4.1. *Let X be separable and let $\mathcal{K}$ and F satisfy (H_2) and (H_3).*

(i) If F is almost ε–δ l.s.c., then (H_6) implies (H_7).

(ii) If F is almost u.s.c., then (H_7) implies (H_6).

Proof. From (H_3) and the fact that X is separable, it follows that there exist a finite or at most countable set Γ, $(\tau_i, \xi_i)_{i\in\Gamma} \subset \mathcal{K}$, $(\rho_i)_{i\in\Gamma} \subset (0, \infty)$, $(\delta_i)_{i\in\Gamma} \subset (0, \infty)$, $(\ell_i)_{i\in\Gamma} \subset L^1(I; \mathbb{R})$ and a negligible set $N_1 \subset I$ such that $\mathcal{K} \subseteq \cup_{i\in\Gamma}(\tau_i - \delta_i, \tau_i + \delta_i) \times S(\xi_i, \rho_i)$ and, for all $i \in \Gamma$, and all $(t, u) \in (((\tau_i - \delta_i, \tau_i + \delta_i) \setminus N_1) \times S(\xi_i, \rho_i)) \cap \mathcal{K}$, we have $\|F(t, u)\| \leq \ell_i(t)$.

We begin with the proof of (i). Since F is ε–δ l.s.c., it follows that, for each $n \in \mathbb{N}$, $n \geq 1$ there exists $I_n \subset I$, with $\lambda(I \setminus I_n) < \frac{1}{n}$, and such that the mapping $(t, \xi) \mapsto F(t, \xi)$ is ε–δ l.s.c. on $(I_n \times X) \cap \mathcal{K}$.

Let $A_n \subset I_n$ the set of all density points of I_n which are also Lebesgue points for ℓ_i for all $i \in \Gamma$. Let $A = (\cup_{n\geq 1} A_n) \cap (I \setminus (N_1 \cup N))$, where N is the negligible set in (H_6). Obviously, $\lambda(I \setminus A) = 0$.

Let $(\tau, \xi) \in (A \times X) \cap \mathcal{K}$. We will show that

$$\liminf_{h\downarrow 0} \frac{1}{h} \mathrm{dist}\left(S(h)\xi + \int_\tau^{\tau+h} S(\tau + h - s)\mathcal{E}_{\tau,\xi,\ell,h}\, ds, K(\tau + h)\right) = 0$$

Let $i_0 \in \Gamma$ and $n_0 \in \mathbb{N}$ such that $\tau \in A_{n_0} \cap (\tau_{i_0} - \delta_{i_0}, \tau_{i_0} + \delta_{i_0})$ and $\xi \in S(\xi_{i_0}, \rho_{i_0})$.From (H_6), it follows that there exists $h_n \downarrow 0$, $f_n \in \mathcal{F}_{F(\tau,\xi)}$ and $p_n \in X$, with $\|p_n\| \to 0$, and such that

$$S(h_n)\xi + \int_\tau^{\tau+h_n} S(\tau + h_n - s)f_n(s)\, ds + h_n p_n \in K(\tau + h_n) \tag{4.1}$$

for all $n \in \mathbb{N}$, $n \geq 1$.

Let $\varepsilon > 0$ be arbitrary but fixed. Since $\mathcal{K}$ is viable by itself there exists $\delta > 0$ and $v \in \mathcal{C}_{\tau,\xi,\ell,\delta}$. Diminishing δ if necessary we may assume that $\tau + \delta < \tau_{i_0} + \delta_{i_0}$ and $v(t) \in S(\xi_{i_0}, \rho_{i_0})$ for all $t \in [\tau, \tau + \delta]$ and

$$F(\tau, \xi) \subset F(t, v(t)) + D(0, \varepsilon) \text{ for all } t \in [\tau, \tau + \delta] \cap A_{n_0}.$$

At this point, let us observe that, for each $n \in \mathbb{N}$, $n \geq 1$, the multi-function $t \mapsto F(t, v(t)) \cap (f_n(t) + D(0, \varepsilon))$ is measurable, nonempty and closed valued from $[\tau, \tau + \delta] \cap A_{n_0}$ to X. Since X is separable, from Kuratowski and Ryll-Nardzewski Theorem 3.1.1, p. 86 in Vrabie [9], it follows that the multi-function above has at least one measurable selection. Let us denote by $g_n : [\tau, T] \cap A_{n_0} \to X$ such a selection. Next, let us extend g_n to a measurable selection of $F(\cdot, v(\cdot))$ on $[\tau, \tau + \delta]$, extension denoted, for simplicity, again by g_n. So, for each $n \in \mathbb{N}$, $n \geq 1$, and $t \in [\tau, \tau + \delta]$, we have

$$g_n(t) \in F(t, v(t)).$$

Also, for each $n \in \mathbb{N}$, $n \geq 1$, and $t \in [\tau, \tau + \delta] \cap A_{n_0}$, we have

$$\|f_n(t) - g_n(t)\| \leq \varepsilon.$$

From (4.1) and the fact that $g_n \in \mathcal{E}_{\tau,\xi,\ell,\delta}$ we deduce that for each $h_n \in (0, \delta)$

$$\frac{1}{h_n}\mathrm{dist}\left(S(h_n)\xi + \int_\tau^{\tau+h_n} S(\tau + h_n - s)\mathcal{E}_{\tau,\xi,\ell,h_n}\, ds, K(\tau + h_n)\right)$$

$$\leq \left\| \frac{1}{h_n} \int_\tau^{\tau+h_n} S(\tau + h_n - s)(g_n(s) - f_n(s))\, ds \right\| + \|p_n\|$$

$$\leq Me^{\omega\delta} \frac{1}{h_n} \int_{[\tau,\tau+h_n]\cap A_{n_0}} \|f_n(s) - g_n(s)\|\, ds$$

$$+ \frac{1}{h_n} \int_{[\tau,\tau+h_n]\setminus A_{n_0}} \|f_n(s) - g_n(s)\|\, ds + \|p_n\|$$

$$\leq Me^{\omega\delta}\varepsilon + \frac{1}{h_n} \int_{[\tau,\tau+h_n]\setminus A_{n_0}} (\|f_n(s)\| + \|g_n(s)\|)\, ds + \|p_n\|$$

$$\leq Me^{\omega\delta}\varepsilon + \frac{1}{h_n} \int_{[\tau,\tau+h_n]\setminus A_{n_0}} (\ell_{i_0}(\tau) + \ell_{i_0}(s))\, ds + \|p_n\|$$

$$\leq Me^{\omega\delta}\varepsilon + \frac{1}{h_n} \int_{[\tau,\tau+h_n]\setminus A_{n_0}} |\ell_{i_0}(s) - \ell_{i_0}(\tau)|\, ds + \frac{2}{h_n} \int_{[\tau,\tau+h_n]\setminus A_{n_0}} \ell_{i_0}(\tau)\, ds + \|p_n\|$$

$$\leq Me^{\omega\delta}\varepsilon + \frac{1}{h_n} \int_\tau^{\tau+h_n} |\, \ell_{i_0}(s) - \ell_{i_0}(\tau)\, |\, ds + 2\ell_{i_0}(\tau) \frac{\lambda([\tau, \tau + h_n] \setminus A_{n_0})}{h_n} + \|p_n\|$$

Passing to lim sup in the inequality above and taking into account that τ is a density point and a Lebesgue point, we get

$$\limsup_{n\to\infty}\frac{1}{h_n}\mathrm{dist}\left(S(h_n)\xi+\int_\tau^{\tau+h_n}S(\tau+h_n-s)\mathcal{E}_{\tau,\xi,\ell,h_n}\,ds,K(\tau+h_n)\right)\le Me^{\omega\delta}\varepsilon$$

and therefore (H_7) holds true and this completes the proof of the first part of Theorem 4.1.

Now let us prove (ii). Since F is almost u.s.c., it follows that for each $n\in\mathbb{N}$, $n\ge 1$ there exists $I_n\subset I$, with $\lambda(I\setminus I_n)<\frac{1}{n}$, such that the mapping $(t,\xi)\mapsto F(t,\xi)$ is u.s.c. on $(I_n\times X)\cap\mathcal{K}$.

Let $A_n\subset I_n$ the set of all density points of I_n which are Lebesgue points too for ℓ_i, for all $i\in\Gamma$. Let $A=(\cup_{n\ge1}A_n)\cap(I\setminus(N_1\cup N))$, where N is the negligible set in (H_7). Obviously, $\lambda(I\setminus A)=0$.

Let $(\tau,\xi)\in\mathcal{K}$. We will show that

$$\liminf_{h\downarrow 0}\frac{1}{h}\mathrm{dist}\left(S(h)\xi+\int_\tau^{\tau+h}S(\tau+h-s)\mathcal{F}_{F(\tau,\xi)}\,ds,K(\tau+h)\right)=0.$$

Let $i_0\in\Gamma$ and $n_0\in\mathbb{N}$ such that $\tau\in A_{n_0}\cap(\tau_{i_0}-\delta_{i_0},\tau_{i_0}+\delta_{i_0})$ and $\xi\in S(\xi_{i_0},\rho_{i_0})$. From (H_7), it follows that there exists $h_n\downarrow 0$, $v_n\in\mathcal{C}_{\tau,\xi,\ell,h_n}$, $f_n\in\mathcal{E}_{\tau,\xi,\ell,h_n}$ and $p_n\in X$, with $\|p_n\|\to 0$, such that for all $n\in\mathbb{N}$, $n\ge 1$ and all $t\in[\tau,\tau+h_n]$ we have $f_n(t)\in F(t,v_n(t))$ and

$$S(h_n)\xi+\int_\tau^{\tau+h_n}S(\tau+h_n-s)f_n(s)\,ds+h_np_n\in K(\tau+h_n)\tag{4.2}$$

Let $\varepsilon>0$ be arbitrary but fixed and let $\delta>0$ be such that

$$F(s,\mu)\subset F(\tau,\xi)+D(0,\varepsilon),\ \text{for all } (s,\mu)\in([\tau,\tau+\delta]\cap A_{n_0}\times S(\xi,\delta))\cap\mathcal{K}$$

Since for all $n\in\mathbb{N},\ n\ge 1$ and all $t\in[\tau,\tau+h_n]$ we have

$$\|v_n(t)-\xi\|\le\|S(t-\tau)\xi-\xi\|+Me^{\omega h_n}\int_\tau^{\tau+h_n}\ell(s)\,ds$$

and diminishing δ, if necessary, we may suppose that $\tau+\delta<\tau_{i_0}+\delta_{i_0}$ and $v_n(t)\in S(\xi_{i_0},\rho_{i_0})\cap S(\xi,\delta)$ for all $n\ge 1$ with $h_n<\delta$ and all $t\in[\tau,\tau+h_n]$. Then, for all $n\ge 1$ with $h_n<\delta$, we get

$$f_n(t)\in F(t,v_n(t))\subset F(\tau,\xi)+D(0,\varepsilon)\text{ for all } t\in[\tau,\tau+h_n]\cap A_{n_0}$$

Using the same arguments as in the first part of the proof we deduce that there exists a measurable selection $g_n : [\tau, \tau + h_n] \cap A_{n_0} \to F(\tau, \xi)$ of the multi-function $t \mapsto F(\tau, \xi) \cap (f_n(t) + D(0, \varepsilon))$ on $[\tau, \tau + h_n] \cap A_{n_0}$. Next, let us extend g_n to $\mathbb{R}$ by using a fixed element in $F(\tau, \xi)$, extension denoted, for simplicity, again by g_n.

From (4.2) and the fact that $g_n \in \mathcal{F}_{F(\tau,\xi)}$ we deduce that for each $h_n \in (0, \delta)$

$$\frac{1}{h_n}\operatorname{dist}\left(S(h_n)\xi + \int_\tau^{\tau+h_n} S(\tau + h_n - s)\mathcal{F}_{F(\tau,\xi)}\, ds, K(\tau + h_n)\right)$$
$$\leq \left\| \frac{1}{h_n} \int_\tau^{\tau+h_n} S(\tau + h_n - s)(g_n(s) - f_n(s))\, ds \right\| + \|p_n\|$$

From now on the proof is identical to the one used in the first part of the Theorem.

$\square$

5 Sufficient conditions for viability

Definition 5.1. We say that the graph $\mathcal{K}$ is:

(i) *locally closed from the left* if for each $(\tau, \xi) \in \mathcal{K}$ there exist $T > \tau$ and $\rho > 0$ such that, for each $(\tau_n, \xi_n) \in ([\tau, T] \times D(\xi, \rho)) \cap \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \widetilde{\tau}$ and $\lim_n \xi_n = \widetilde{\xi}$, we have $(\widetilde{\tau}, \widetilde{\xi}) \in \mathcal{K}$;

(ii) *closed from the left* if for each $(\tau_n, \xi_n) \in \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \widetilde{\tau}$ and $\lim_n \xi_n = \widetilde{\xi}$, we have $(\widetilde{\tau}, \widetilde{\xi}) \in \mathcal{K}$;

(iii) *locally compact from the left* if, it is locally closed from the left and, for each $(\tau, \xi) \in \mathcal{K}$ there exist $T > \tau$ and $\rho > 0$ such that, for each $(\tau_n, \xi_n) \in ([\tau, T] \times D(\xi, \rho)) \cap \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, and $\lim_n \tau_n = \widetilde{\tau}$, there exists a convergent subsequence $(\xi_{n_k})_k$ of $(\xi_n)_n$;

(iv) *compact from the left* if, it is closed from the left and, for each $(\tau_n, \xi_n) \in \mathcal{K}$ with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \widetilde{\tau}$, and $(\xi_n)_n$ bounded, there exists a convergent subsequence $(\xi_{n_k})_k$ of $(\xi_n)_n$.

Remark 5.1. Let $(\xi_{n_k})_k$ be the subsequence of $(\xi_n)_n$ whose existence is ensured by (ii) in Definition 5.1 and let $\xi = \lim_k \xi_{n_k}$. Then $(\tau, \xi) \in \mathcal{K}$.

Definition 5.2. By a *Carathéodory uniqueness function* we mean a function $\alpha : I \times R_+ \to R_+$ such that:

(i) for each $x \in \mathbb{R}_+$, $t \mapsto \alpha(t, x)$ is locally integrable;

(ii) for a.e. $t \in I$, $x \mapsto \alpha(t, x)$ is continuous, nondecreasing;

(iii) for each $\tau \in I$, the only absolutely continuous solution of the Cauchy problem
$$\begin{cases} x'(t) = \alpha(t, x(t)) \\ x(\tau) = 0 \end{cases}$$
is $x \equiv 0$.

Definition 5.3. We say that $A + F$ is β-compact if for all $(\tau, \xi) \in \mathcal{K}$ there exists $\delta > 0$, $\rho > 0$, a Carathéodory uniqueness function, $\alpha : I \times R_+ \to R_+$, a negligible set $N \subset I$ and a continuous function $m : [0, \infty) \to [0, \infty)$, such that, for all $B \subseteq D(\xi, \rho)$, all $t \in (0, \infty)$ and all $s \in [\tau - \delta, \tau + \delta] \setminus N$ we have
$$\beta(S(t)F((\{s\} \times B) \cap \mathcal{K})) \leq m(t)\alpha(s, \beta(B)). \tag{5.1}$$

Remark 5.2.

(i) If the C_0-semigroup $\{S(t);\ t \geq 0\}$ is compact and F is integrally bounded then $A + F$ is β-compact.

(ii) If F is β-compact (see definition 5.3 in Popescu [8]), then $A + F$ is β-compact.

Theorem 5.1. *Let $\mathcal{K}$ be locally closed from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (H_2), (H_3) and (H_5) are satisfied and $A + F$ is β-compact then a necessary and sufficient condition in order that $\mathcal{K}$ be mild viable with respect to $A + F$ is (H_7).*

Theorem 5.2. *Let $\mathcal{K}$ be locally compact from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (H_2), (H_3) and (H_5) are satisfied, then a necessary and sufficient condition in order that $\mathcal{K}$ be viable with respect to $A + F$ is (H_7).*

From Theorems 5.1, 5.2 and Brezis-Browder Ordering Principle, i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [3], we easily deduce the two global viability results stated below.

Theorem 5.3. *Let $\mathcal{K}$ be closed from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (H_2), (H_3) and (H_5) are satisfied and $A + F$ is β-compact then a necessary and sufficient condition in order that $\mathcal{K}$ be globally mild viable with respect to $A + F$ is (H_7).*

Theorem 5.4. *Let $\mathcal{K}$ be compact from the left and let $F : \mathcal{K} \rightsquigarrow X$ be nonempty, convex and weakly compact valued. If (H_2), (H_3) and (H_5) are satisfied, then a necessary and sufficient condition in order that $\mathcal{K}$ be globally viable with respect to $A + F$ is (H_7).*

The next lemma, essentially inspired from Cârjă, Monteiro-Marques [1], is the main step through the proof of both Theorems 5.1 and 5.2.

Lemma 5.1. *Let I be a nonempty and bounded interval and $K : I \rightsquigarrow X$ a multi-function with locally closed from the left graph, $\mathcal{K}$, let $(\tau, \xi) \in \mathcal{K}$ and let $F : \mathcal{K} \rightsquigarrow X$ be a nonempty valued multi-function. Suppose (H_1), (H_2), (H_3) and (H_7) are satisfied. Let $Z \subseteq I$ be a negligible set including the negligible set in (H_7) and $\ell \in L^1(I, \mathbb{R})$ be the function from the definition of $\mathcal{E}_{\tau,\xi,\ell,h}$.*

Let $\rho > 0$ and $T > \tau$ be such that:

(1) $([\tau, T] \times D(\xi, \rho)) \cap \mathcal{K}$ *is closed from the left*;

(2) $\|F(t, u)\| \leq \ell(t)$ *a.e. for* $t \in [\tau, T]$ *and for all* $u \in K(t) \cap D(\xi, \rho)$;

(3) T *and* ρ *satisfy Definition* 2.5;

(4) $\sup_{t \in [\tau,T]} \|S(t-\tau)\xi - \xi\| + Me^{\omega(T-\tau)} \int_\tau^T l(s)ds + Me^{\omega(T-\tau)}(T-\tau) < \rho.$

Then, for each $\varepsilon \in (0, 1)$ and each open set $\mathcal{O} \subseteq I$, with $Z \subseteq \mathcal{O}$, there exist a family $\mathcal{P}_T = \{[t_m, s_m); m \in \Gamma\}$, of disjoint intervals, with Γ finite or at most countable, and five functions $f, r, v \in L^1(\tau, T; X)$, $\theta : \{(t, s);\ \tau \leq s \leq t \leq T\} \to [0, T - \tau]$ measurable, and $u \in C([\tau, T]; X)$ such that:

(i) $\cup[t_m, s_m) = [\tau, T)$ *and* $s_m - t_m \leq \varepsilon$, *for all* $m \in \Gamma$;

(ii) *if* $t_m \in \mathcal{O}$, *then* $[t_m, s_m) \subseteq \mathcal{O}$;

(iii) $u(t_m) \in D(\xi, \rho) \cap K(t_m)$, *for all* $m \in \Gamma$, $u(T) \in D(\xi, \rho) \cap K(T)$;

(iv) $\theta(t, s) \leq t - s$; $t \mapsto \theta(t, s)$ *nonexpansive on* $(s, T]$ *and, for each* $t \in (\tau, T]$, $s \mapsto \theta(t, s)$ *measurable on* $[\tau, t)$;

(v) $v \in C([t_m, s_m); X)$; $(t, v(t)) \in ([\tau, T) \times S(\xi, \rho)) \cap \mathcal{K}$ *for all* $t \in [\tau, T)$ *and* $\|v(t) - u(t_m)\| \le \varepsilon$ *for all* $t \in [t_m, s_m)$;

(vi) $f(s) \in F(s, v(s))$ *a.e. for* $s \in [t_m, s_m)$ *if* $t_m \notin \mathcal{O}$ *and* $\|f(s)\| \le l(s)$ *a.e. for* $s \in [\tau, T]$;

(vii) $\|r(s)\| \le \varepsilon$ *a.e. for* $s \in [\tau, T]$;

(viii) $u(t) = S(t-\tau)\xi + \int_\tau^t S(t-s)f(s)ds + \int_\tau^t S(\theta(t,s))r(s)ds$ *for all* $t \in [\tau, T]$;

(ix) $\|u(t) - u(t_m)\| \le \varepsilon$ *for all* $t \in [t_m, s_m)$ *and* $m \in \Gamma$.

Proof. Let ε be arbitrary but fixed in $(0,1)$ and let $\mathcal{O} \subseteq R$ be an open subset with $Z \subseteq \mathcal{O}$. We will show that there exist $\delta = \delta(\varepsilon, \mathcal{O}) \in (\tau, T)$ and $\mathcal{P}_\delta, f, r, v, \theta, u$ such that (i)$\sim$(ix) hold true with δ instead of T. We distinguish between the following different cases.

Case 1. If $\tau \in \mathcal{O}$, we take $\Gamma = \{1\}$, $t_1 = \tau$, $s_1 = \delta$ with $\delta \in (\tau, T)$ small enough in order to $[\tau, \delta) \subseteq \mathcal{O}$, $\tau - \delta \le \varepsilon$ and there exists a simple solution (f, v) issuing from (τ, ξ), defined on $[\tau, \delta]$ with $\|f(s)\| \le \ell(s)$ a.e. for $s \in [\tau, \delta]$. Further, let us diminish δ such that $\|v(t) - \xi\| < \min\{\varepsilon, \rho\}$ for all $t \in [\tau, \delta]$ and let us define $\mathcal{P}_\delta = \{[\tau, \delta)\}$, $\theta = 0$, $r = 0$ and $u(t) = v(t)$ for all $t \in [\tau, \delta]$.

Case 2. If $\tau \notin \mathcal{O}$ then $\tau \notin Z$ which implies that there exist $h_n \downarrow 0$, $v_n \in \mathcal{C}_{\tau,\xi,\ell,h_n}$, $f_n \in \mathcal{E}_{\tau,\xi,\ell,h_n}$ such that $f_n(s) \in F(s, v_n(s))$ a.e. for $s \in [\tau, \tau + h_n]$ and $p_n \in X$, with $\|p_n\| \to 0$, such that

$$S(h_n)\xi + \int_\tau^{\tau+h_n} S(\tau + h_n - s) f_n(s)\, ds + p_n h_n \in K(\tau + h_n)$$

for all $n \in \mathbb{N}$, $n \ge 1$. Let $n_0 \in N$ and $\delta = \tau + h_{n_0}$ be such that $\delta \in (\tau, T)$, $h_{n_0} < \varepsilon$, $\|p_{n_0}\| < \varepsilon$ and

$$\sup_{t \in [\tau, \tau + h_{n_0}]} \|S(t-\tau)\xi - \xi\| + M e^{\omega h_{n_0}} \int_\tau^{\tau + h_{n_0}} \ell(s)\, ds + h_{n_0} \le \min\{\varepsilon, \rho\}$$

We define $\mathcal{P}_\delta = \{[\tau, \delta)\}$, $f(s) = f_{n_0}(s)$, $\theta(t,s) = 0$ for $\tau \le s \le t \le \delta$, $r(s) = p_{n_0}$, $v(s) = v_{n_0}(s)$ for $s \in [\tau, \delta]$, and let $u : [\tau, \delta] \to X$ be given by (viii). We may easily see that (i)$\sim$(ix) are satisfied.

Let

$$\mathcal{U} = \{(\mathcal{P}_\delta, f, r, v, \theta, u);\ \delta \in (\tau, T],\ \text{(i)}\sim\text{(ix) hold true with } \delta \text{ instead of } T\}.$$

As we already have shown, $\mathcal{U} \neq \emptyset$. On $\mathcal{U}$ we define a partial order by:

$$(\mathcal{P}_{\delta_1}, f_1, r_1, v_1, \theta_1, u_1) \preceq (\mathcal{P}_{\delta_2}, f_2, r_2, v_2, \theta_2, u_2),$$

if

$$\begin{cases} \delta_1 \leq \delta_2, \quad \mathcal{P}_{\delta_1} \subseteq \mathcal{P}_{\delta_2}, \\ f_1(s) = f_2(s), r_1(s) = r_2(s), v_1(s) = v_2(s) \text{ a.e. for } s \in [\tau, \delta_1] \\ \theta_1(t,s) = \theta_2(t,s) \text{ for } \tau \leq s \leq t \leq \delta_1 \\ u_1(s) = u_2(s), \text{for all } s \in [\tau, \delta_1]. \end{cases}$$

We will prove that each nondecreasing sequence in $\mathcal{U}$ is bounded from above. Let $(\mathcal{P}_{\delta_j}, f_j, r_j, v_j, \theta_j, u_j)_{j\geq 1}$ be a nondecreasing sequence in $\mathcal{U}$ and let $\delta = \sup_{j\geq 1}\delta_j$. If there exists $j_0 \in N$ such that $\delta_{j_0} = \delta$, then $(\mathcal{P}_{\delta_{j_0}}, f_{j_0}, r_{j_0}, v_{j_0}, \theta_{j_0}, u_{j_0})$ is an upper bound for the sequence. So, let us assume that $\delta_j < \delta$, for all $j \geq 1$. Obviously, $\delta \in (\tau, T]$. We define $\mathcal{P}_\delta = \cup_{j\geq 1}\mathcal{P}_{\delta_j}$, $f(s) = f_j(s)$, $\theta(t,s) = \theta_j(t,s)$ for $\tau \leq s \leq t \leq \delta_j$, $v(s) = v_j(s)$ and $r(s) = r_j(s)$ for all j and all $s \in [\tau, \delta_j)$. Clearly, $f, r, v \in L^1(\tau, \delta; X)$. Since $|\theta_j(\delta_j, s) - \theta_i(\delta_i, s)| \leq |\delta_j - \delta_i|$ for all $i, j \geq 1$ and $\tau \leq s < \min\{\delta_i, \delta_j\}$, we may define $\theta(\delta, s) = \lim_{j\to\infty} \theta_j(\delta_j, s)$ for all $\tau \leq s < \delta$. One may easily see that θ satisfies (iv). Next, we define $u : [\tau, \delta] \to X$ by

$$u(t) = S(t-\tau)\xi + \int_\tau^t S(t-s)f(s)\,ds + \int_\tau^t S(\theta(t,s))r(s)\,ds,$$

for all $t \in [\tau, \delta]$. We have $u \in C([\tau, \delta]; X)$ and $u(s) = u_j(s)$, for all $j \geq 1$ and all $s \in [\tau, \delta_j]$. Since $u(\delta) = \lim_{t\uparrow\delta} u(t) = \lim_{j\to\infty} u(\delta_j) = \lim_{j\to\infty} u_j(\delta_j)$, and $u_j(\delta_j) \in D(\xi, \rho) \cap K(\delta_j)$ and the latter is closed from the left, we deduce that $u(\delta) \in D(\xi, \rho) \cap K(\delta)$. The rest of conditions in lemma being obviously satisfied, it follows that $(\mathcal{P}_\delta, f, r, v, \theta, u)$ is an upper bound for the sequence. Thus, the partially ordered set $(\mathcal{U}, \preceq)$ and the function $\mathcal{N} : (\mathcal{U}, \preceq) \to R$, defined by $\mathcal{N}(\mathcal{P}_\delta, f, r, v, \theta, u) = \delta$, for each $(\mathcal{P}_\delta, f, r, v, \theta, u) \in \mathcal{U}$, satisfy the hypotheses of the Brezis-Browder Ordering Principle, i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [3]. Accordingly, there exists an $\mathcal{N}$-maximal element in $\mathcal{U}$. This means that there exists $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) \in \mathcal{U}$ such that, whenever

$$(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) \preceq (\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{\theta}, \overline{u}),$$

we necessarily have

$$\mathcal{N}(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) = \mathcal{N}(\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{\theta}, \overline{u}).$$

We will show that $\delta^* = T$. To this aim, let us assume by contradiction that $\delta^* < T$. We distinguish between two cases.

Case 1. If $\delta^* \in \mathcal{O}$, we take $\overline{\delta} \in (\delta^*, T)$ such that $[\delta^*, \overline{\delta}] \subseteq \mathcal{O}$ and $\overline{\delta} - \delta^* < \varepsilon$ and there exists a simple solution (g, v) issuing from $(\delta^*, u^*(\delta^*))$ defined on $[\delta^*, \overline{\delta}]$ with $\|g(s)\| \leq \ell(s)$ a.e. for $s \in [\delta^*, \overline{\delta}]$. We may diminish $\overline{\delta}$ such that $\|v(t) - u^*(\delta^*)\| \leq \varepsilon$ for all $t \in [\delta^*, \overline{\delta}]$. Let us define

$$\overline{f}(s) = \begin{cases} f^*(s) \text{ for } s \in [\tau, \delta^*] \\ g(s) \text{ a.e for } s \in (\delta^*, \overline{\delta}] \end{cases}, \overline{r}(s) = \begin{cases} r^*(s) \text{ for } s \in [\tau, \delta^*] \\ 0 \text{ for } s \in (\delta^*, \overline{\delta}] \end{cases},$$

$$\overline{\theta}(t,s) = \begin{cases} \theta^*(t,s) & \text{for } \tau \leq s \leq t \leq \delta^* \\ t - \delta^* + \theta^*(\delta^*, s) & \text{for } \tau \leq s < \delta^* < t < \overline{\delta} \\ 0 & \text{for } \delta^* \leq s < t \leq \overline{\delta} \end{cases},$$

$$\overline{v}(s) = \begin{cases} v^*(s), \text{ for } s \in [\tau, \delta^*) \\ v(s), \text{ for } s \in [\delta^*, \overline{\delta}] \end{cases}, \quad \overline{u}(s) = \begin{cases} u^*(s), \text{ for } s \in [\tau, \delta^*] \\ v(s), \text{ for } s \in (\delta^*, \overline{\delta}] \end{cases}$$

and $\mathcal{P}_{\overline{\delta}} = \mathcal{P}_{\delta^*} \cup \{[\delta^*, \overline{\delta})\}$.

It follows that $(\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{\theta}, \overline{u}) \in \mathcal{U}$, $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) \preceq (\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{\theta}, \overline{u})$, but $\delta^* < \overline{\delta}$ which contradicts the maximality of $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*)$.

Case 2. If $\delta^* \notin \mathcal{O}$ then $\delta^* \notin Z$ which implies that there exist $h_n \downarrow 0$, $v_n \in \mathcal{C}_{\delta^*, u^*(\delta^*), \ell^*, h_n}$, $f_n \in \mathcal{E}_{\delta^*, u^*(\delta^*), \ell^*, h_n}$ such that $f_n(s) \in F(s, v_n(s))$ a.e. for $s \in [\delta^*, \delta^* + h_n]$ and $p_n \in X$, with $\|p_n\| \to 0$, such that

$$S(h_n)u^*(\delta^*) + \int_{\delta^*}^{\delta^* + h_n} S(\delta^* + h_n - s) f_n(s)\, ds + p_n h_n \in K(\delta^* + h_n)$$

for all $n \in \mathbb{N}$, $n \geq 1$. Since by (4) in Lemma 5.1 $u^*(\delta^*) \in S(\xi, \rho)$ we may choose $n_0 \in \mathbb{N}$ and $\overline{\delta} = \delta^* + h_{n_0}$ be such that $\overline{\delta} \in (\tau, T)$, $h_{n_0} < \varepsilon$, $\|p_{n_0}\| < \varepsilon$ and

$$\sup_{t \in [\delta^*, \delta^* + h_{n_0}]} \|S(t - \delta^*)u^*(\delta^*) - u^*(\delta^*)\| + M e^{\omega h_{n_0}} \int_{\delta^*}^{\delta^* + h_{n_0}} \ell^*(s)\, ds + M e^{\omega(T - \tau)} h_{n_0} \leq \nu$$

where $\nu = \min\{\varepsilon, \rho - \|u^*(\delta^*) - \xi\|\}$.

Let us define $\mathcal{P}_{\overline{\delta}} = \mathcal{P}_{\delta^*} \cup \{[\delta^*, \overline{\delta})\}$, $\overline{\theta}$ as in **Case 1** and

$$\overline{f}(s) = \begin{cases} f^*(s), s \in [\tau, \delta^*] \\ f_{n_0}(s), s \in (\delta^*, \overline{\delta}] \end{cases}, \overline{r}(s) = \begin{cases} r^*(s), s \in [\tau, \delta^*] \\ p_{n_0}, s \in (\delta^*, \overline{\delta}] \end{cases}, \overline{v}(s) = \begin{cases} v^*(s), s \in [\tau, \delta^*] \\ v_{n_0}, s \in (\delta^*, \overline{\delta}] \end{cases},$$

$$\overline{u}(t) = \begin{cases} u^*(t), t \in [\tau, \delta^*] \\ S(t-\delta^*)u^*(\delta^*) + \int_{\delta^*}^t S(t-s) f_{n_0}(s)\, ds + (t-\delta^*)p_{n_0}, \text{for } t \in (\delta^*, \overline{\delta}]. \end{cases}$$

We can easily see that (i)∼(ix) are satisfied. So, $(\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{\theta}, \overline{u}) \in \mathcal{U}$ and, in addition, $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*) \preceq (\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{\theta}, \overline{u})$. But $\delta^* < \overline{\delta}$ which contradicts the maximality of $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, \theta^*, u^*)$. Hence $\delta^* = T$, and $\mathcal{P}_{\delta^*}$, f^*, r^*, v^*, θ^* and u^* satisfy all the conditions (i)∼(ix). The proof is complete.

□

Definition 5.4. Let $\varepsilon > 0$, Z and $\mathcal{O}$ be as in Lemma 5.1. An element $(\mathcal{P}_T, f, r, v, \theta, u)$ satisfying (i)∼(ix) in Lemma 5.1, is called an $(\varepsilon, \mathcal{O})$*-approximate solution of* (1.1).

6 Proof of Theorems 5.1 and 5.2

Proof. Since the necessity follows from Theorem 3.1, we will confine ourselves only to the proof of the sufficiency.

Let $Z \subseteq \mathbb{R}$ be a negligible set including the negligible sets appearing in (H_7) and Definition 5.3. Let $\varepsilon_n \in (0,1)$, with $\varepsilon_n \downarrow 0$, let $(\mathcal{O}_n)_{n\geq 1} \subseteq \mathbb{R}$ be a sequence of open sets, and let ℓ the function in Lemma 5.1. We notice that we may assume with no loss of generality that the sequence $(\mathcal{O}_n)_{n\geq 1}$ is so chosen to satisfy:

- (a) $Z \subseteq \mathcal{O}_n$ for each $n \in \mathbb{N}$, $n \geq 1$;
- (b) $\mathcal{O}_{n+1} \subseteq \mathcal{O}_n$ and $\lambda([\tau, T] \cap \mathcal{O}_n) \leq \varepsilon_n$ for each $n \in \mathbb{N}$, $n \geq 1$;
- (c) $F_{|[(I \setminus \mathcal{O}_n) \times D(\xi, \rho)] \cap \mathcal{K}}$ is strongly-weakly u.s.c., for each $n \in \mathbb{N}$, $n \geq 1$;

Let $\rho > 0$ and $T > \tau$ be as in Lemma 5.1, and such that ρ satisfies Definition 5.3 and let $n \in \mathbb{N}$, $n \geq 1$ be arbitrary but fixed. Let $((\mathcal{P}_T^n, f_n, r_n, v_n, \theta_n, u_n))_n$ be a sequence of $(\varepsilon_n, \mathcal{O}_n)$-approximate solutions of (1.1), sequence whose existence is ensured, again by Lemma 5.1. If $\mathcal{P}_T^n = \{[t_m^n, s_m^n);\ m \in \Gamma_n\}$ with Γ_n finite or at most countable, we denote by $a_n : [\tau, T) \to [\tau, T)$ the step function, defined by $a_n(s) = t_m^n$ for each $s \in [t_m^n, s_m^n)$. Clearly

$$\lim_n a_n(s) = s \tag{6.1}$$

uniformly for $s \in [\tau, T)$.

We will show that, on a subsequence at least, $(u_n)_n$ is uniformly convergent on $[\tau, T]$ to some function u.

We analyze first the case when X is separable. From (vii) in Lemma 5.1, it follows that, for each $t \in [\tau, T]$, we have

$$\beta\left(\left\{\int_\tau^t S(\theta_n(t,s))r_n(s)\,ds;\; n \geq 1\right\}\right) = 0. \tag{6.2}$$

Next, let us observe that

$$\|f_n(t)\| \leq \ell(t) \tag{6.3}$$

for each $n \geq 1$ and a.e for $t \in [\tau, T]$.

From (v) and (ix), we deduce that

$$\lim_n \|u_n(a_n(s)) - u_n(s)\| = 0 \text{ and } \lim_n \|u_n(a_n(s)) - v_n(s)\| = 0$$

uniformly for $s \in [\tau, T)$. So we have $\lim_n \|v_n(s) - u_n(s)\| = 0$ uniformly for $s \in [\tau, T)$. Then

$$\beta(\{v_n(s) - u_n(s);\; n \geq 1\}) = 0 \tag{6.4}$$

for each $s \in [\tau, T)$.

Next, by (viii) in Lemma 5.1, we obtain

$$u_n(t) = S(t-\tau)\xi + \int_\tau^t S(t-s)f_n(s)\,ds + \int_\tau^t S(\theta_n(t,s))r_n(s)\,ds \tag{6.5}$$

for all $n \geq 1$ and $t \in [\tau, T]$.

Let $k \in \mathbb{N}$, $k \geq 1$ and $t \in [\tau, T]$. In view of (6.2), (6.5) and Lemma 2.1, we have

$$\begin{aligned} &\beta(\{u_n(t);\; n \geq k\}) \\ &\leq \beta\left(\left\{\int_\tau^t S(t-s)f_n(s)\,ds;\; n \geq k\right\}\right) + \beta\left(\left\{\int_\tau^t S(\theta_n(t,s))r_n(s)\,ds;\; n \geq k\right\}\right) \\ &\leq \int_{[\tau,t]\setminus \mathcal{O}_k} \beta(\{S(t-s)f_n(s);\; n \geq k\})\,ds + \int_{\mathcal{O}_k} \beta(\{S(t-s)f_n(s);\; n \geq k\})\,ds \end{aligned} \tag{6.6}$$

Since $f_n(s) \in F(s, v_n(s))$ a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$ and $A + F$ is β-compact we deduce that

$$\beta(\{S(t-s)f_n(s);\; n \geq k\}) \leq m(t-s)\alpha(s, \beta(\{v_n(s);\; n \geq k\}))$$

for all $t \in [\tau, T]$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. Let $\alpha_0 = (\sup_{s\in[0,T-\tau]} m(s))\alpha$, then α_0 is a Carathéodory uniqueness function, too.

So, from (6.6) and (6.3) it follows that

$$\beta(\{u_n(t);\ n \geq k\}) \leq \int_{[\tau,t]\setminus\mathcal{O}_k} \alpha_0(s, \beta(\{v_n(s);\ n \geq k\})\, ds + Me^{\omega(T-\tau)} \int_{\mathcal{O}_k} \ell(s)\, ds$$

Since by (6.4) we have $\beta(\{u_n(t);\ n \geq k\}) = \beta(\{v_n(t);\ n \geq k\})$ and $\beta(\{u_n(t);\ n \geq k\}) = \beta(\{u_n(t);\ n \geq 1\})$, passing to the limit for $k \to \infty$ in the inequality above and taking into account that α_0 is a Carathéodory uniqueness function, it follows that $\beta(\{u_n(t);\ n \geq 1\}) = 0$. Thus $\{u_n(t);\ n \geq 1\}$ is relatively compact for each $t \in [\tau, T]$. In view of (6.3) and using (6.2) and Theorem 8.4.1, p. 194 in Vrabie [10] we conclude that, on a subsequence at least, $(u_n)_n$ is uniformly convergent on $[\tau, T]$ to some function u. But $\lim_n v_n(t) = u(t)$, uniformly for $t \in [\tau, T)$, and hence, for each $k \geq 1$, the set

$$C_k = \overline{\{(t, v_n(t));\ n \geq k,\ t \in [\tau, T) \setminus \mathcal{O}_k\}}$$

is compact. Since F is strongly-weakly u.s.c. and has weakly compact values, by Lemma 2.6.1, p. 47, in Cârjă, Necula, Vrabie [3], it follows that, for each $k \geq 1$, the set

$$B_k := \overline{\text{conv}} \left(\bigcup_{n\geq k} \bigcup_{t\in[\tau,T]\setminus\mathcal{O}_k} F(t, v_n(t)) \right)$$

is weakly compact. We notice that $\|f_n(s)\| \leq \ell(s)$ a.e. for $s \in [\tau, T]$ and $f_n(s) \in B_k$ for each $k \geq 1$ and $n \geq k$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. Since $\ell \in L^1(\tau, T; \mathbb{R})$, B_k is weakly compact and $\lim_k \lambda(\mathcal{O}_k) = 0$, by Diestel's Theorem 1.3.8, p. 10, in Cârjă, Necula, Vrabie [3], it follows that, on a subsequence at least, $\lim_n f_n = f$ weakly in $L^1(\tau, T; X)$. As $\lim_n v_n(t) = u(t)$ uniformly for $t \in [\tau, T]$, and, by Lemma 5.1, for each $k \geq 1$, each $n \geq k$, we have $f_n(s) \in F(s, v_n(s))$ a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$, from Theorem 3.1.2, p. 88, in Vrabie [9], we conclude that $f(s) \in F(s, u(s))$ for each $k \geq 1$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. Since $\lim_k \lambda(\mathcal{O}_k) = 0$, we get

$$f(s) \in F(s, u(s)) \text{ a.e. for } s \in [\tau, T] \tag{6.7}$$

Finally, passing to the limit both sides in (6.5), for $n \to \infty$, we get

$$u(t) = S(t-\tau)\xi + \int_\tau^t S(t-s)f(s)\, ds,$$

for each $t \in [\tau, T]$. Since $v_n(t) \in K(t)$ and $\lim_n v_n(t) = u(t)$ for all $t \in [\tau, T)$ and $\mathcal{K}$ is locally closed from the left, it follows that $u(t) \in K(t)$ for each $t \in [\tau, T]$. By (6.7), we conclude that u is a mild solution of (1.1), and this completes the proof when X is separable.

If X is not separable, we have to observe that there exists a separable and closed subspace $Y \subseteq X$ such that the families: $\{S(\cdot)f_n(\cdot);\ n \geq 1\}$, $\{S(\cdot)u_n(\cdot);\ n \geq 1\}$, $\{S(\cdot)v_n(\cdot);\ n \geq 1\}$ and $\{S(\cdot)r_n(\cdot);\ n \geq 1\}$ are Y-valued. Then, to complete the proof, it suffices to follows the very same arguments as before and to make use of (iv) in Remark 2.1.

The proof of Theorem 5.2 is exactly the same with the exception of obtaining the fact that $\{u_n(t);\ n \geq 1\}$ is relatively compact. Indeed, since $\mathcal{K}$ is locally compact from the left, it follows that the set $\{v_n(t); n \geq 1\}$ is relatively compact. Moreover, recalling that

$$\lim_n \|v_n(s) - u_n(s)\| = 0$$

for $s \in [\tau, T)$, it follows that $\{u_n(t);\ n \geq 1\}$ is relatively compact for all $t \in [\tau, T)$. The remaining of the proof is identical to the one of Theorem 5.1. $\square$

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DYNAMIC ANALYSIS OF TWO ADHESIVELY BONDED RODS*

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Abstract

This work presents two models for the dynamic analysis of two rods that are adhesively bonded. The first model assumes that the adhesive is an elasto-plastic material and that complete debonding occurs when the stress reaches the yield limit. In the second model the degradation of the adhesive is described by the introduction of material damage. Failure occurs when the material is completely damaged, or the damage reaches a critical floor value. Both models are analyzed and the existence of a weak solution is established for the model with damage. In the quasistatic case, a new condition for adhesion is found as the limit of the adhesive thickness tends to zero.

keywords: Adhesion, elastic rod, dynamic contact, bonding function, existence and uniqueness

1 Introduction

We study two different models for the dynamic process of debonding of two slender rods that are adhesively bonded. In the first model, the adhesive

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is treated as a short rod made of a softer elasto-plastic material. System failure, i.e., complete debonding, occurs when the stress reaches the yield limit of the adhesive material. In the second model, the adhesive is treated as a damageable rod via the use of a damage function. In this case, there is a continuous decrease in the adhesive strength as cycles of tension and compression progress. The adhesive undergoes cumulative damage, similar to fatigue, and may completely fail, even if the cyclic stress never reaches the yield limit.

There is considerable interest in the engineering literature in models for material damage and metal fatigue, since predicting damage failure is of paramount concern to the design engineer.

Recent mathematical models for material damage, following the fundamental idea of Kachanov in the 1960s (see [11] for details) of introducing an internal variable, the *damage function* that measures the damage of the material, can be found in the monographs [10, 18, 22, 25], as well as in the recent papers [6, 7, 13, 17] and in the references therein. The various aspects of general models of material damage were studied in these references. Models of damage in specialized settings, similar to the one in this paper, can be found in [2, 3, 4]. Related mathematical models are those of adhesion, where a surface internal variable, the *bonding function*, was introduced by Frémond [10] and has a similar interpretation, namely, it measures the damage of the surface bonds.

Mathematical models for adhesive contact can be found in the monographs [22, 25] and in recent papers [1, 8, 9, 15, 20, 21] (see also the references therein).

In this paper we combine the two concepts of a damage function and a bonding function, and use the first to derive the source function for the debonding process. We consider a simplified one-dimensional model of two rods glued together. In this model we obtain an evolution equation for the bonding function by considering the evolution of the damage of the glue as the glue layer becomes relatively thin.

This work is the continuation of [21], where the quasistatic model was studied and numerically simulated. However, there the model did not allow for complete debonding in finite time. Models which allow complete debonding can be found in [15, 20] and here. We note that some of the models proposed and used in the above literature do not allow for complete debonding, and the issue is under current study.

As noted above, we consider a setting in which two thin rods are glued, and the glue is considered as a third (shorter) rod. In one of the models, the adhesive layer is considered as a damageable material. System failure happens when the adhesive reaches complete damage, and then the rods completely debond. The main interest in this work is in the models, and in the limit when the thickness of the adhesive layer approaches zero.

We present the two dynamic models in Section 2: one without, and the other one with material damage. We establish the existence of a weak solution for the second model in Section 4, and obtain interesting estimates on the strain in Section 5. For the first model the existence of the unique solution is straightforward to show. Then, in Section 3, we study the quasistatic problem, which reduces to a nonlinear ordinary differential equation for the damage function, since the equations of motion for the displacements can be integrated. Thus, we obtain expressions for the time to failure, i.e., the time to complete debonding. We also pass to the limit when the glue thickness is very small, and obtain an evolution equation for the adhesive as a limit of the damage equation, Problem $P_{\zeta 0}$. In this way, we obtain a new expression (unlike any in the above references) for the debonding source function, in the limit of the damage source function. This is the main modeling novelty in the paper. Some of the estimates in Section 5 are new, too.

The paper concludes with Section 6, where some future research suggestions can be found.

2 The model

Figure 1 depicts the setting of the two bonded rods. The left end of the first rod is attached to a movable device. The reference configuration of the rods are $0 \le x \le l_1$ and $l_2 \le x \le L$ ($l_1 < l_2$), and the interval $[l_1, l_2]$ is occupied by the adhesive, assumed to be a softer deformable material.

The horizontal displacements of the rods are $u_i = u_i(x,t)$, where $i = 1, 2$ for rod 1 and rod 2, respectively. The displacement of the adhesive is $u_0 = u_0(x,t)$. Below, we use the subscripts 1 and 2 for the rods, and 0 for the adhesive.

We are also interested in the limit case when the thickness of the adhesive layer vanishes, i.e., $|l_2 - l_1| \to 0$.

A body force of density $f_B = f_B(x,t)$ (per unit length) is acting on the rods, and on the adhesive segment. The left end ($x = 0$) of rod 1 is subjected

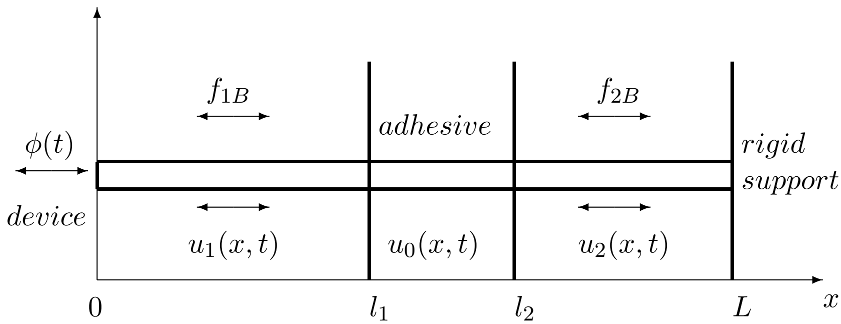

Figure 1. Two rods in adhesive contact

to a dynamic axial displacement $\phi = \phi(t)$. Thus $u_1(0,t) = \phi(t)$. The right end of rod 2 is fixed, so $u_2(L,t) = 0$. When ϕ is negative, the rods are in tension, and when ϕ is positive, the are in compression.

The dynamic motion of each one of the three rods is described by the wave equation and the displacements are assumed to be continuous at the interfaces $x = l_1, l_2$ where the tractions are equal, too.

We consider two different scenarios, which result in two different models. In the first scenario, the adhesive is considered as an elasto-plastic material with lower modulus of elasticity, as compared to the rods. The adhesion between the two rods is assumed to break down, or completely debond, when the stress in the adhesive region reaches the yield limit.

In the second model we assume that the adhesive material undergoes damage as a result of the strains. Then, complete debonding occurs when the damage reaches the threshold limit.

We denote by ρ_i and E_i, for $i = 0, 1, 2$, the density (per unit length) and the elasticity modulus of the material in each region.

The classical formulation of the first model for *the vibrations of two rods in adhesive contact* is:

Problem P_{cl}. Find a triple of functions (u_1, u_0, u_2) such that, for $0 < t \le T$:

$$\rho_1 u_{1tt}(x,t) - E_1 u_{1xx}(x,t) = \rho_1 f_B(x,t), \qquad x \in (0, l_1), \tag{1}$$
$$\rho_0 u_{0tt}(x,t) - E_0 u_{0xx}(x,t) = \rho_0 f_B(x,t), \qquad x \in (l_1, l_2), \tag{2}$$
$$\rho_2 u_{2tt}(x,t) - E_2 u_{2xx}(x,t) = \rho_2 f_B(x,t), \qquad x \in (l_2, L), \tag{3}$$
$$u_1(0,t) = \phi(t), \qquad u_2(L,t) = 0, \tag{4}$$
$$u_1(l_1,t) = u_0(l_1,t), \qquad E_1 u_{1x}(l_1,t) = E_0 u_{0x}(l_1,t), \tag{5}$$
$$u_2(l_2,t) = u_0(l_2,t), \qquad E_2 u_{2x}(l_2,t) = E_0 u_{0x}(l_2,t), \tag{6}$$
$$u(x,0) = u_{in}(x), \tag{7}$$
$$u_t(x,0) = v_{in}(x). \tag{8}$$

Here, u_{in} and v_{in} are the (prescribed) initial displacements and velocities, respectively, with the understanding that $u_1(x,0) = u_{in}(x)$ and $u_{1t}(x,0) = v_{in}(x)$ for $x \in [0, l_1]$, and similarly for the other two rods.

The problem consists of three coupled wave equations for the displacements $u_1(x,t), u_2(x,t)$, and $u_0(x,t)$.

To describe the second model, we follow [11] (see also [10, 18, 22, 25] and the references therein) and introduce the *damage function* $\zeta = \zeta(x,t)$, which measures the pointwise fractional decrease in the strength of the adhesive material. To describe the damage process of the material the damage-free adhesive modulus of elasticity E_0 is replaced with the effective modulus

$$E_{eff} = \zeta E_0.$$

Then, it follows that

$$0 \le \zeta(x,t) \le 1, \tag{9}$$

and when $\zeta = 1$ the material is damage-free; when $\zeta = 0$ the damage is complete and the system breaks at the point; and when $0 < \zeta(x,t) < 1$ the material is partially damaged and has a decreased load carrying capacity.

Next, we need to describe the evolution of the damage function ζ. Following [10, 11, 22, 25] (see also the other references mentioned above), we assume that the evolution of damage is caused by the growth of micro-cracks and micro-cavities caused by the cyclic stress. The damage function has to satisfy the growth equation

$$\zeta_t - \kappa \zeta_{xx} = \Phi(\zeta, u_{0x}) + \xi,$$

where, $\Phi = \Phi(\zeta, u_{0x})$ is the *damage source function*, which is described shortly in (10), κ is the damage diffusion coefficient, and ξ is a 'force' that prevents ζ from violating (9). To describe the latter, we let $I_{[0,1]}$ denote the indicator function of the interval $[0,1]$, and then its subdifferential is the set-valued mapping denoteb by $\partial I_{[0,1]}(z)$. To enforce the condition $0 \leq \zeta \leq 1$, we require that $-\xi \in \partial I_{[0,1]}(\zeta)$. Indeed, when $0 < \zeta < 1$ then $\xi = 0$; when $\zeta = 0$ then $\xi > 0$ has the exact value that prevents ζ from becoming negative; and when $\zeta = 1$ then $\xi < 0$ has the exact value that prevents ζ from exceeding the value one.

General damage source functions can be found in [10, 22, 25]; here, we use a somewhat simple function which depends only on the mechanical energy $E_0\zeta u_{0x}^2$ and the damage process is assumed to be irreversible so that once micro-cavities or micro-crack are formed, they do not mend, thus

$$\Phi(\zeta, u_{ax}) = -d(\zeta u_{0x}^2 - \epsilon_0)_+. \tag{10}$$

Here, d is the damage rate coefficient, ϵ_0 is the scaled damage threshold energy, below which there is no damage change, and $(r)_+$ is the positive part function, i.e., $(r)_+ = r$ if $0 \leq r$ and $(r)_+ = 0$ if $r < 0$. The negative sign makes the process irreversible. With this choice, the parabolic equation for ζ (with $\xi = 0$) predicts that if initially $\zeta_{in} \leq 1$, then $\zeta \leq 1$ for $0 < t$.

For the sake of generality, we also assume that the adhesive has viscosity which we model with $\nu(\zeta u_{0tx})_x$, where ν is the viscosity coefficient, assumed to be small.

Problem P_ζ. Find a quadruple of functions (u_1, u_0, ζ, u_2) such that, for $0 < t \leq T$ (1), (3), (4), (7), and (8) hold, together with

$$\rho_0 u_{0tt}(x,t) - E_0(\zeta u_{0x})_x(x,t) - \nu(\zeta u_{0tx})_x(x,t) = \rho_0 f_B(x,t),\ x \in (l_1, l_2), \tag{11}$$

$$\zeta_t - \kappa\zeta_{xx} + d(\zeta u_{0x}^2 - \epsilon_0)_+ \in -\partial I_{[0,1]}(\zeta), \qquad x \in (l_1, l_2), \tag{12}$$

$$u_1(l_1,t) = u_0(l_1,t), \quad E_1 u_{1x}(l_1,t) = E_0(\zeta u_{0x})(l_1,t), \tag{13}$$

$$u_2(l_2,t) = u_0(l_2,t), \quad E_2 u_{2x}(l_2,t) = E_0(\zeta u_{0x})(l_2,t), \tag{14}$$

$$\zeta_x(l_1,t) = 0 = \zeta_x(l_2,t), \quad \zeta(x,0) = \zeta_{in}(x). \tag{15}$$

Here, ζ_{in} is the initial damage, which has the value one in a damage-free material.

The analysis of problems P_{cl} and P_ζ will be done in Section 4. Next, we study the equations for the problems when the process is quasistatic and the adhesive layer is thin.

3 Quasistatic problems

We study three problems which model the process when it is quasistatic, i.e., slow enough so that the acceleration terms may be neglected, and in the absence of body forces ($f_B = 0$).

3.1 Quasistatic version of P_{cl}

We begin with the quasistatic version of Problem P_{cl}. Since there are no body forces and the second time derivatives are neglected, the displacements are linear. Writing

$$u_0(x,t) = \alpha(t)x + \beta(t), \tag{16}$$

straightforward manipulations, using the facts that the displacements u_1 and u_2 are linear and the boundary conditions (4)–(6), yield

$$\alpha(t) = \frac{-\phi(t)}{(l_2 - l_1) + \frac{E_0}{E_2}(L - l_2) + \frac{E_0}{E_1} l_1}, \tag{17}$$

and

$$\beta(t) = -\alpha \left(l_2 + (L - l_2)\frac{E_0}{E_2} \right). \tag{18}$$

Moreover,

$$u_1(x,t) = \frac{E_0}{E_1}\alpha(t)x + \phi(t), \qquad u_2(x,t) = -\frac{E_0}{E_2}\alpha(t)(L - x). \tag{19}$$

We note that when the displacement ϕ is negative the system is under tension and when it is positive the system is under compression.

In the limit when the thickness of the layer of glue tends to zero, $l_2 \to l_1 = l$, we find that

$$\alpha(t) = \frac{-\phi(t)}{\frac{E_0}{E_2}(L - l) + \frac{E_0}{E_1} l}, \qquad \beta(t) = -\alpha\left(l + (L - l)\frac{E_0}{E_2} \right).$$

Thus, the influence of the adhesive enters via its stiffness E_0. The displacement at $x = l$ is given by

$$u_1(l,t) = u_2(l,t) = \frac{\phi(t)(L - l)E_1}{E_1(L - l) + E_2 l}.$$

The stress by $p(t) = E_1 u_{1x}(l,t) = E_0 \alpha(t) = E_2 u_{2x}(l,t)$. Therefore, this system will debond (completely) only when the stress reaches the plasticity yield or the debonding limit σ^*,

$$E_0 \alpha(t) = \sigma^*.$$

Clearly, this formulation cannot take into account gradual degradation of the strength of the bonds as a result of cycles in ϕ.

The quasistatic problem with a prescribe traction boundary condition at $x = 0$ is straightforward to study, and is not very interesting, since in a one-dimensional system the stress is uniform.

3.2 Quasistatic version of P_ζ

We turn to the quasistatic version of Problem P_ζ, which turns out to be more interesting. In particular, it accounts for degradation of the strength of the bonds as a result of cycles in ϕ. Since there are no body forces and the second time derivatives are neglected, the displacements u_1 and u_2 are linear. In equation (11) for u_0 we neglect the viscosity term, and obtain $(\zeta u_{0x})_x = 0$. Therefore,

$$\zeta(x,t) u_{0x}(x,t) = \gamma(t), \qquad l_1 \le x \le l_2, \tag{20}$$

where $\gamma(t)$ is to be determined. Then, the boundary conditions (13) and (14) yield

$$E_1 u_{1x}(l_1,t) = E_0 \gamma, \qquad E_2 u_{2x}(l_2,t) = E_0 \gamma.$$

Thus,

$$u_{1x}(l_1,t) = \frac{E_0}{E_1}\gamma(t), \qquad u_{2x}(l_2,t) = \frac{E_0}{E_2}\gamma(t),$$

and then,

$$u_1(x,t) = \frac{E_0}{E_1}\gamma(t)x + \phi(t), \qquad u_2(x,t) = -\frac{E_0}{E_2}\gamma(t)(L-x).$$

Next, integration in (20) yields

$$u_0(x,t) = \gamma(t) \int_{l_1}^{x} \frac{1}{\zeta(x,t)}\, dx + \delta(t), \tag{21}$$

for $l_1 \leq x \leq l_2$, where δ is a constant of integration. It follows from the continuity of the displacements that

$$u_1(l_1,t) = \frac{E_0}{E_1}\gamma(t)l_1 + \phi(t) = \delta(t),$$

$$u_2(l_2,t) = -\frac{E_0}{E_2}\gamma(t)(L-l_2) = \gamma(t)\int_{l_1}^{l_2}\frac{1}{\zeta(x,t)}\,dx + \delta(t).$$

Let

$$c_{12} = \frac{E_0}{E_2}(L-l_2) + \frac{E_0}{E_1}l_1.$$

Substituting δ from the first equation and rearranging yields

$$\gamma(t) = \frac{-\phi(t)}{c_{12} + \int_{l_1}^{l_2}\frac{1}{\zeta(x,t)}\,dx}. \tag{22}$$

Then,

$$\delta(t) = \frac{E_0}{E_1}\gamma(t)l_1 + \phi(t) = \phi(t) - \frac{E_0 l_1 \phi(t)}{E_1 c_{12} + E_1\int_{l_1}^{l_2}\frac{1}{\zeta(x,t)}\,dx}. \tag{23}$$

It follows that once ζ is found, the problem is solved. To obtain ζ, we note that $u_{0x} = \gamma/\zeta$, hence

$$\Phi(u_{ax}) = -d(\zeta u_{ax}^2 - \epsilon_0)_+ = -d\left(\frac{\gamma^2}{\zeta} - \epsilon_0\right)_+ = -d\left(\Theta(\phi;\zeta,t) - \epsilon_0\right)_+,$$

where we defined

$$\Theta(\phi;\zeta,t) = \frac{\phi^2(t)}{\zeta\left(c_{12} + \int_{l_1}^{l_2}\frac{1}{\zeta(x,t)}\,dx\right)^2}.$$

Now, the problem for ζ is the following.

Problem $P_{quas-\zeta}$. Given ϕ, find a function $\zeta = \zeta(x,t)$ such that, for $0 < t \leq T$,

$$\zeta_t - \kappa\zeta_{xx} = -d(\Theta(\phi;\zeta,t) - \epsilon_0)_+, \quad x \in (l_1,l_2), \tag{24}$$

$$\zeta(x,0) = \zeta_{in}, \quad \zeta_x(l_1,t) = \zeta_x(l_2,t) = 0. \tag{25}$$

We note that the problem is nonlocal, since the source term on the right-hand side of (24) depends on $\int_{l_1}^{l_2} \frac{1}{\zeta(x,t)}\,dx$. It is somewhat unusual and has mathematical interest in and of itself, and will be analyzed elsewhere.

Next, we consider the limit $\lim l_1 = \lim l_2 = l$. It follows from the boundary conditions (25) that $\zeta = \zeta(t)$ only, as it does not depend on x. Also,

$$\lim_{|l_2-l_1|\to 0} \Theta(\phi;\zeta,t) = \Theta_0(\zeta,t) = \frac{\phi^2(t)}{c_{12}^2\zeta}.$$

Therefore, the limit problem is as follows.

Problem $P_{\zeta 0}$. Find a function $\zeta = \zeta(t)$ such that, for $0 < t \le T$,

$$\zeta' = -d\left(\frac{\phi^2(\tau)}{c_{12}^2\zeta} - \epsilon_0\right)_+, \tag{26}$$

$$\zeta(0) = \zeta_{in}. \tag{27}$$

The problem is a nonlinear ordinary differential equation with non-Lipschitz right-hand side. We study it in Section 4.

We note that when ϵ_0 is negligible, as compared to the average of $\phi^2(t)/c_{12}^2$, the equation for ζ becomes

$$\zeta' = -d\Theta_0(\zeta,t) = -\frac{d\phi^2(t)}{c_{12}^2\zeta}.$$

Using the initial condition, we obtain

$$\zeta^2(t) = \zeta_{in}^2 - \frac{2d}{c_{12}^2}\int_0^t \phi^2(\tau)\,d\tau.$$

It follows that the time to failure t_0^* is given in this case implicitly by

$$\int_0^{t_0^*} \phi^2(\tau)\,d\tau = \frac{c_{12}^2\zeta_{in}^2}{2d}.$$

A simple comparison argument shows that if t^* is the time to failure of the solution of (26) and (27), then $t_0^* \le t^*$, as one would expect.

Problem $P_{\zeta 0}$ connects material damage and adhesion at the joint point and it has a very different structure from the usual bonding conditions used in the literature (see, e.g., [21]). Indeed, there, the bonding was assumed to be of the form

$$\zeta' = -d\zeta(u_x^2 - \epsilon_0)_+,$$

which doesn't allow for failure, i.e., complete debonding in finite time, or a more recent condition ([15])

$$\zeta' = -d\zeta^{\alpha}(u_x^2 - \epsilon_0)_+,$$

which allows for failure when $0 \le \alpha < 1$. Here, we find that $\alpha = -2$, and this makes the analysis quite different.

3.3 Quasistatic version of P_ζ with traction condition

We describe briefly the case when instead of the displacement ϕ, a traction $q = q(t)$ is applied at the left end ($x = 0$). This is often the case in experimental settings. Thus, we replace the first condition in (4) with $E_1 u_{1x}(0,t) = q(t)$. Then,

$$u_1(x,t) = \frac{1}{E_1} q(t)x + b(t),$$

where $b(t)$ is to be determined. At $x = l_1$ we have $E_1 u_{1x}(l_1,t) = q(t) = E_0\gamma(t)$, hence

$$\gamma(t) = \frac{1}{E_0} q(t).$$

Moreover, $u_2(x,t) = (q(t)/E_2)(x - L)$. It follows from (21) that

$$u_0(x,t) = \frac{1}{E_0} q(t) \int_{l_1}^{x} \frac{1}{\zeta(x,t)}\, dx + \delta(t), \tag{28}$$

for $l_1 \le x \le l_2$, and δ is a constant. The displacements' continuity implies

$$\frac{1}{E_1} q(t) l_1 + b(t) = \delta(t), \qquad \frac{1}{E_0} q(t) \int_{l_1}^{l_2} \frac{1}{\zeta(x,t)}\, dx + \delta(t) = \frac{1}{E_2} q(t)(l_2 - L).$$

It follows that

$$\delta(t) = -q(t) \left(\frac{1}{E_2}(L - l_2) + \frac{1}{E_0} \int_{l_1}^{l_2} \frac{1}{\zeta(x,t)}\, dx \right). \tag{29}$$

Also,

$$b(t) = -q(t) \left(\frac{1}{E_1} l_1 + \frac{1}{E_2}(L - l_2) + \frac{1}{E_0} \int_{l_1}^{l_2} \frac{1}{\zeta(x,t)}\, dx \right). \tag{30}$$

It is seen that once ζ is found, the displacements u_1, u_2, and u_0 are given by the expressions above. It remains to obtain an equation for ζ. We have

$$\Phi(u_{0x}) = -d(\zeta u_{0x}^2 - \epsilon_0)_+ = -d\left(\frac{q^2(t)}{E_0^2\zeta} - \epsilon_0\right)_+ .$$

We conclude that the quasistatic problem for ζ, when a traction q is prescribed at $x = 0$, is the following.

Problem $P_{\zeta q}$. Given $q(t)$, find a function $\zeta = \zeta(x, t)$ such that, for $0 < t \le T$,

$$\zeta_t - \kappa\zeta_{xx} = -d\left(\frac{q^2(t)}{E_0^2\zeta} - \epsilon_0\right)_+, \quad x \in (l_1, l_2), \tag{31}$$

$$\zeta(x, 0) = \zeta_{in}, \quad \zeta_x(l_1, t) = \zeta_x(l_2, t) = 0. \tag{32}$$

We note that this problem is local, but is also somewhat unusual and has mathematical interest in and of itself, and will be analyzed elsewhere.

The problem for a thin layer of glue is obtained in the limit $\lim l_2 = l_1 = l$.

Problem $P_{\zeta q0}$. Given $q(t)$, find a function $\zeta = \zeta(t)$ such that, for $0 < t \le T$,

$$\zeta' = -d\left(\frac{q^2(t)}{E_0^2\zeta} - \epsilon_0\right)_+, \tag{33}$$

$$\zeta(0) = \zeta_{in}. \tag{34}$$

We note that whereas problems $P_{quas-\zeta}$ and $P_{\zeta q}$ are substantially different, the limit problems $P_{\zeta 0}$ and $P_{\zeta q0}$ are very similar, with q^2/E_0^2 replacing ϕ^2/c_{12}^2. Therefore, the existence of the unique solution of Problem $P_{\zeta q0}$ follows from Theorem 1 below.

In this case, if we neglect the Dupré energy ϵ_0, we find that the time to complete debonding t_0^* is given implicitly by

$$\int_0^{t_0^*} q^2(\tau)\, d\tau = \frac{E_0^2\zeta_0^2}{2d}.$$

4 Analysis

We first study Problem $P_{\zeta 0}$, (26) and (27), and establish the existence of a unique local (in time) solution. Then, we prove the existence of a weak solution to the dynamic problem with damage, Problem P_ζ.

4.1 Problem $P_{\zeta 0}$

For the sake of generality, we replace the function $\phi^2(t)/c_{12}^2$ in (26) with a more general nonnegative smooth and bounded function $\psi = \psi(t)$. Then, the problem is as follows.

Problem $P_{\zeta\psi}$. Given a function ψ, find a function $\zeta = \zeta(t) \geq 0$, such that, for $0 < t \leq T$,

$$\zeta' = -d\left(\frac{\psi(t)}{\zeta} - \epsilon_0\right)_+, \tag{35}$$

$$\zeta(0) = \zeta_{in}. \tag{36}$$

We make the following assumptions on the problem data.

H_1. The function $\psi : [0, T] \to [0, \infty)$ is continuous and bounded.
H_2. The constants d and ϵ_0 are positive and $\zeta_{in} \in (0, 1]$.

Theorem 1. *Assume that H_1 and H_2 hold. Then there exists $T^* > 0$ such that there exists a unique solution ζ of Problem $P_{\zeta\psi}$ on the time interval $[0, T^*)$. Moreover,*

$$\zeta \in C^1([0, T^*)). \tag{37}$$

Proof. Let $0 < a < \zeta_{in}$ and let $g_a(\zeta, t)$ be a function with the graph of a straight line through $(0, 0)$ and $-d\left(\frac{\psi(t)}{a} - \epsilon_0\right)$, and let

$$F(\zeta, t) \equiv \max\left(-d\left(\frac{\psi(t)}{\zeta} - \epsilon_0\right)_+, \; g_a(\zeta, t)\right).$$

Then, $F(\zeta, t)$ is Lipschitz in ζ and so there exists a unique solution to

$$\zeta' = F(\zeta, t), \quad \zeta(0) = \zeta_{in}.$$

Letting t_a^* be the value of t at which $\zeta(t)$ first equals a, then, since $0 < a$ is arbitrary, the theorem follows when we choose $T^* = \sup(t_a^*)$, for $a \in (0, \zeta_{in})$.

4.2 Problem P_ζ

We turn to Problem P_ζ, and establish the existence of its weak solution. The weak formulation is obtained in the usual manner, and we use the following notation: u represents the displacements, and is such that $u = u_1$ on $[0, l_1]$, $u = u_0$ on $[l_1, l_2]$, and $u = u_2$ on $[l_2, L]$. Similarly, we define the functions

$\rho(x)$ and $c(x)$ as $\rho = \rho_1, c = E_1$ on $[0, l_1]$, $\rho = \rho_0, c = E_0$ on $[l_1, l_2]$, and $\rho = \rho_2, c = E_2$ on $[l_2, L]$. Finally, for the sake of generality we add a viscosity term in (1) and (3), and let the viscosity $\nu(x)$ be defined in the same way. We also extend the definition of the unknown function ζ as 1 outside of the interval $[l_1, l_2]$, and replace c with $c\zeta$ in (1)–(3).

We now multiply equations (1)–(3) by a test function φ, integrate by parts and use the boundary conditions to obtain the following weak formulation for u, for a.a. $t \in (0, T)$,

$$\int_0^L \rho(x) u_{tt}(x,t)\varphi(x)\,dx + \int_0^L c(x)\zeta(x,t)u_x(x,t)\varphi_{x(}(x)\,dx$$

$$+ \int_0^L \nu(x)\zeta(x,t)u_{xt}(x,t)\varphi_{x(}(x)\,dx = \int_0^L \rho(x) f_B(x,t)\varphi(x)\,dx.$$

Similarly, using θ as a test function, we obtain from (12),

$$\int_{l_1}^{l_2} \zeta_t(x,t)\theta(x)\,dx + \kappa \int_{l_1}^{l_2} \zeta_x(x,t)\theta_x(x)\,dx$$

$$\geq -d \int_{l_1}^{l_2} (\zeta(x,t)u_x^2(x,t) - \epsilon_0)_+ \theta(x)\,dx.$$

Actually, as explained below, we can eliminate the subgradient term because the source term for damage is sufficient to keep the damage parameter in the interval of interest.

We regard the adhesive and the two rods as a single continuum, as described above, but damage is assumed to affect only the adhesive.

To proceed with the analysis we need the following spaces.

$$V \equiv H_0^1(0, L), \quad H \equiv L^2(0, L),$$

and

$$\mathcal{V} \equiv \mathcal{L}^2(0, T; V), \quad \mathcal{H} \equiv \mathcal{L}^2(0, T; H).$$

We use on V and $\mathcal{V}$ the (equivalent) norms

$$||w||_V^2 = \int_0^L w_x^2 ds, \qquad ||w||_{\mathcal{V}}^2 = \int_0^T \int_0^L w_x^2\, dsdt.$$

We need to introduce a truncation to preserve the coercivity of the problem, which becomes noncoercive in the limit $\zeta \to 0$. To that end we let η

be a truncation function, assumed to be smooth and nondecreasing with the following properties:

$$\eta(r) \leq 2 \;\text{ if } r \geq 1, \quad \eta(r) = \delta \;\text{ if } r < \delta, \quad \eta(r) = r \;\text{ if } r \in (2\delta, 1],$$

where δ is assumed to be very small, in particular, $\delta << \varepsilon_0$. We note that these problems, typically, possess only local solutions, so this is not a serious restriction. Moreover, we show below that η is not active (i.e., $\eta(\zeta) = \zeta$) on some interval of time.

Now, we define the operator $A : V \to V'$ as follows: for ζ in H let

$$\langle A(\zeta, u), v \rangle \equiv \int_0^L c(x)\, \eta(\zeta(x))\, u_x(x)\, v_x(x)\, dx.$$

We note that $c(x)$ is discontinuous, and bounded away from zero, as it takes the values E_1, E_0, E_2 in the different intervals. We also assume that

$$\phi \in C^2([0, T]).$$

To obtain homogeneous boundary conditions at $x = 0, x = L$ we define a new variable $w(x,t) = u(x,t) - \phi(t)(1 - x/L)$ and obtain a similar equation for w involving only a change in $f_B(x,t)$, but with w satisfying zero boundary conditions at $x = 0$ and $x = L$. Therefore, we assume at the outset that $\phi(t) = 0$ to make the presentation simpler. To slightly simplify the presentation we also assume that the density $\rho(x)$ is a constant, rescaled as $\rho = 1$. In addition, we let

$$v(t) \equiv u'(t), \quad v(t) \in \mathcal{V}, \quad u(t) \equiv u_0 + \int_0^t v(s)\, ds.$$

The truncated problem is as follows. Find $v \in \mathcal{V}$ such that,

$$
\begin{aligned}
v' + A(\zeta, v) + A(\zeta, u) &= f, && (38)\\
v(0) &= v_0, && (39)\\
u(t) &\equiv u_0 + \int_0^t v(s)\, ds, \; u_0 \in V. && (40)
\end{aligned}
$$

Here f is a body force, assumed in $\mathcal{H}$. The problem for the damage is to find $\zeta \in W^{1,2}((0, L) \times (0, T))$ such that,

$$
\begin{aligned}
\zeta' - \Delta\zeta &= -d\left(\eta(\zeta)\, \mathcal{X}_{[l_1, l_2]} Q_M(u_x) - \varepsilon_0\right)_+, && (41)\\
\zeta(0) &= \zeta_0 \in H^1(0, L), \quad \zeta_0(x) \in (3\delta, 1]. && (42)
\end{aligned}
$$

We let $\zeta_0(x) = 1$ for $x \notin [l_1, l_2]$. This forces the extension of ζ to the rest of $[0, L]$ to equal 1. Then, the requirement $\zeta \in H^1(0, L)$ guarantees that $\zeta = 1$ at the end points $x = l_1, l_2$, so damage is happening in the interior of this interval but not at the ends. Also, we obtain the natural boundary conditions $\zeta_x = 0$ at the endpoints of the adhesion interval.

Moreover, $Q_M(r)$ is a truncation of u_x, making it easier to obtain some of the estimates below. It is a bounded Lipschitz continuous function which equals r^2 whenever $|r| < M$, say

$$Q_M \in C^1(R), \qquad 0 \leq Q(r) \leq M^2. \tag{43}$$

The characteristic function $\mathcal{X}_{[l_1, l_2]}$ of the middle interval is used to guarantee that the damage process is taking place only in the glue layer.

We show below that on a suitable interval the truncation is inactive but, to begin with, it is convenient to include it. The source term for damage in (41) is such that together with the assumptions on ζ_0, it implies that $\zeta(x, t) \in (\delta, 1]$ a.e. x for all t. It is a consequence of maximum principle arguments and a proof can be found in [13].

We begin with the study of the mechanical part of the problem.

Lemma 1. *Let $\zeta \in \mathcal{H}$. Then there exists a unique solution to* (38) – (40)*. Also, if v_ζ is the solution corresponding to ζ then the map $\zeta \to v_\zeta$ is continuous from $\mathcal{H}$ to $\mathcal{V}$.*

Proof. We consider the existence part first. It follows from standard theorems in Lions, [19], that there exists a unique solution v_u to (38) for each $u \in \mathcal{V}$. Also, the operator $Av(t) \equiv A(\zeta(t), v(t))$ is monotone, hemicontinuous, bounded, and coercive as a map from $\mathcal{V}$ to $\mathcal{V}'$, so the the main existence theorem in [16] is applicable. Consider now the map $\Psi : \mathcal{V} \to \mathcal{V}$, given by

$$\Psi(u(t)) \equiv u_0 + \int_0^t v_u(s)\, ds.$$

Then,

$$\Psi(u(t)) - \Psi(w(t)) = \int_0^t (v_u(s) - v_w(s))\, ds.$$

Next, simple manipulations, using (38), yield

$$\frac{1}{2}\|v_u(t) - v_w(t)\|_H^2 + \frac{\delta}{2} a \int_0^t ||v_u - v_w||_V^2\, ds \leq C_\delta \int_0^t ||w - u||_V^2\, ds.$$

It follows that

$$\begin{aligned}\|\Psi(u\,(t)) - \Psi(w\,(t))\|_V^2 &\leq C_T \int_0^t ||v_u\,(s) - v_w\,(s)||_V^2\, ds \\ &\leq C_T C_\delta \int_0^t ||u\,(s) - w\,(s)||_V^2\, ds,\end{aligned}$$

and this implies that a large enough power of Ψ is a contraction mapping on $\mathcal{V}$, so there exists a unique solution (v, u) to (38)–(40).

Let (v, u) be a solution of this initial value problem. Then, it follows from the equation that

$$\frac{1}{2}\|v\,(t)\,\|_H^2 + \frac{\delta}{2} a \int_0^t ||v||_V^2\, ds$$

$$\leq \frac{1}{2}\|v_0\|_H^2 + C_\delta \int_0^t ||u||_V^2\, ds + C\,(f) + \int_0^t \|v\|_H^2 ds$$

$$\leq C_{\delta T} \int_0^t \int_0^s ||v||^2\, dr ds + C\,(f, ||u_0||_V) + \int_0^t \|v\|_H^2 ds.$$

Here and below, we denote by $C = C(\cdots)$ a constant that depends only on the argument and the problem constants. It follows from Gronwall's inequality that there exists a constant, depending on the indicated quantities, such that

$$\|v\,(t)\,\|_H^2 + \int_0^t ||v||_V^2\, ds \leq C\left(|v_0|_H^2\,, f, ||u_0||_V\,, \delta\right). \tag{44}$$

Next, we show the continuous dependence of the solution (v, u) on ζ. Let v_i correspond to $\zeta_i, i = 1, 2$. Then, from the initial value problem (38)–(40), together with routine manipulations, we obtain

$$\frac{1}{2}\|v_1\,(t) - v_2\,(t)\,\|_H^2 + \frac{\delta}{2} a \int_0^t ||v_1\,(s) - v_2\,(s)||_V^2\, ds$$

$$\leq C_\delta \int_0^t \int_\Omega |\eta\,(\zeta_1) - \eta\,(\zeta_2)|^2\, |v_{1x}|^2\, dx ds$$

$$+ C_\delta \int_0^t \int_\Omega |\eta\,(\zeta_1) - \eta\,(\zeta_2)|^2\, |u_{1x}|^2\, dx ds. \tag{45}$$

Assume that the map $\zeta \to v_\zeta$ is not continuous. Then, there exists $\zeta \in \mathcal{H}$ and a sequence $\{\zeta_n\}$ such that $\zeta_n \to \zeta$ pointwise, as well as in $\mathcal{H}$, but for some $\varepsilon > 0$,

$$\int_0^T ||v_n(s) - v(s)||_V^2 \, ds \geq \varepsilon,$$

where v is the solution of (38)–(40) that corresponds to ζ and v_n corresponds to ζ_n. Now, let $t = T$ and $v_2 = v_n, v_1 = v$ in (45). Since η is a bounded function, the dominated convergence theorem applies and the right-hand side of (45) converges to zero, which is a contradiction. This proves the lemma.

The next two theorems are used below, and can be found in Lions [19] and Simon [24], respectively.

Theorem 2. *Assume $p \geq 1, q > 1$, and $W \subseteq U \subseteq Y$, where the inclusion map $W \to U$ is compact and the inclusion map $U \to Y$ is continuous. Let*

$$S_R = \{\mathbf{u} \in L^p(0,T;W) : \mathbf{u}' \in L^q(0,T;Y),\ ||\mathbf{u}||_{L^p(0,T;W)} + ||\mathbf{u}'||_{L^q(0,T;Y)} < R\}.$$

Then S_R is precompact in $L^p(0,T;U)$.

Theorem 3. *Let W, U, and Y be as in Theorem 2, $q > 1$, and let*

$$S_{RT} = \{\mathbf{u} : ||\mathbf{u}(t)||_W + ||\mathbf{u}'||_{L^q(0,T;Y)} \leq R, \quad t \in [0,T]\}.$$

Then S_{RT} is precompact in $C(0,T;U)$.

We now consider the question of existence for a solution (v, ζ) of (38) –(42). To that end let $\zeta \in \mathcal{H}$ be given. Then, let (v_ζ, u_ζ) denote the unique solution of problem (38)–(40). Using ζ and u_ζ in the right side of (41) and (42), it follows from a well known results of Brezis ([5]), see also Showalter ([23]), since the differential operator $-\Delta$ is a subgradient of a proper lower semicontinuous functional, that there exists a unique function $\xi \in L^2\left(0,T;H^2(0,L)\right), \xi' \in \mathcal{H}, \zeta_x = 0$ at $x = 0$ and L, which satisfies (41) and (42). Let $\Phi(\zeta) \equiv \xi$. Thus, this Φ is a map from $\mathcal{H}$ to $\mathcal{H}$. It was shown in Lemma 1 that the map $\zeta \to v_\zeta$ is continuous from $\mathcal{H}$ to $\mathcal{V}$. From the definition of u_ζ as an integral of v_ζ given in (40), it follows that $\zeta \to u_\zeta$ is continuous from $\mathcal{H}$ to $C([0,T];V)$. Therefore, since all the truncation functions in the source term for damage in (41) are bounded and Lipschitz continuous, it follows from simple manipulations, such as those above, that

$\zeta \to \Phi(\zeta)$ is continuous as a map from $\mathcal{H}$ to $\mathcal{H}$. In fact, more can be said, but this is enough for our purposes. We note the fact that Φ is not only continuous, but maps $\mathcal{H}$ into a compact subset of $\mathcal{H}$. This follows from Theorem 2 and the following interesting lemma which is stated in more generality than needed here.

Lemma 2. *Assume that the boundary of Ω is in $C^{1,1}$. Let $y, y' \in L^2(0,T; L^2(\Omega))$, $y(0) = y_0 \in H^1(\Omega)$, assume also that $y \in L^2\left(0,T; H^2(\Omega)\right)$ and it satisfies $\partial y/\partial n = 0$ on $\partial\Omega$. Then,*

$$\int_0^t \left(y', -\Delta y\right)_{L^2(\Omega)} ds = \frac{1}{2}\|\nabla y(t)\|^2_{L^2(\Omega)^d} - \frac{1}{2}\|\nabla y_0\|^2_{L^2(\Omega)^d}.$$

Proof. Let $Ly \equiv -\Delta y$, where $y \in D(L)$ is given by

$$\left\{y \in L^2\left(0,T; L^2(\Omega)\right); \Delta y \in L^2\left(0,T; L^2(\Omega)\right),\ \partial y/\partial n = 0 \text{ on } \partial\Omega\right\}.$$

Then, L is a maximal monotone operator. Also, since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, it follows that $D(L)$ is dense in $L^2\left(0,T; L^2(\Omega)\right)$. Let

$$y_\varepsilon \equiv (I + \varepsilon L)^{-1} y,$$

for a small $\varepsilon > 0$. Thus, $y'_\varepsilon = (I + \varepsilon L)^{-1} y' \in D(L)$ and it is routine to verify that

$$\int_0^t \left(y'_\varepsilon, (-\Delta y_\varepsilon)\right)_{L^2(\Omega)} ds = \frac{1}{2}\|\nabla y_\varepsilon(t)\|^2_{L^2(\Omega)^d} - \frac{1}{2}\|\nabla y_\varepsilon(0)\|^2_{L^2(\Omega)^d}.$$

Moreover, since $D(L)$ is dense in $L^2\left(0,T; L^2(\Omega)\right)$, it follows from standard results on maximal monotone operators (see, e.g., [5]) that, as $\varepsilon \to 0$,

$$\begin{aligned} -\Delta y_\varepsilon = Ly_\varepsilon = L(I+\varepsilon L)^{-1} y &= (I+\varepsilon L)^{-1} Ly \to Ly = -\Delta y, \\ (I+\varepsilon L)^{-1} y' &= y'_\varepsilon \to y' \quad \text{in } L^2\left(0,T; L^2(\Omega)\right). \end{aligned}$$

In addition,

$$\begin{aligned} \nabla y_\varepsilon &= \nabla (I+\varepsilon L)^{-1} y = (I+\varepsilon L)^{-1} \nabla y \to \nabla y, \\ \nabla y_\varepsilon(0) &= \nabla (I+\varepsilon L)^{-1} y_0 = (I+\varepsilon L)^{-1} \nabla y_0 \to \nabla y_0, \end{aligned}$$

and by using subsequences, if necessary, all these convergence results take place for a.a. t. Therefore, for a.a. t,

$$\begin{aligned}
& \frac{1}{2}\|\nabla y(t)\|^2_{L^2(\Omega)^d} - \frac{1}{2}\|\nabla y_0\|^2_{L^2(\Omega)^d} \\
= & \lim_{\varepsilon\to 0}\frac{1}{2}\|\nabla y_\varepsilon(t)\|^2_{L^2(\Omega)^d} - \frac{1}{2}\|\nabla y_\varepsilon(0)\|^2_{L^2(\Omega)^d} \\
= & \lim_{\varepsilon\to 0}\int_0^t \left(y_\varepsilon', -\Delta y_\varepsilon\right)_{L^2(\Omega)} ds = \int_0^t \left(y', -\Delta y\right)_{L^2(\Omega)} ds.
\end{aligned}$$

Now, using the fact the source term for damage in (41) is bounded independently of ζ and u_x, it follows from the lemma that

$$\frac{1}{2}\|\zeta_x(t)\|^2_H + \frac{1}{2}\int_0^t \|\Delta\zeta(s)\|^2_H ds \le \frac{1}{2}\|\zeta_{0x}\|^2_H + C(M).$$

This estimate, along with (41), shows that ζ' is bounded in $\mathcal{H}$. Thus, we obtain an estimate of the form,

$$\|\zeta'\|^2_{\mathcal{H}} + \frac{1}{2}\|\zeta_x(t)\|^2_H + \frac{1}{2}\int_0^t \|\Delta\zeta(s)\|^2_H ds \le \frac{1}{2}\|\nabla\zeta_0\|^2_H + C(M).$$

Using now Theorem 3, it follows that the image $\Phi(\mathcal{H})$ belongs to a compact subset of $C([0,T];U) \subseteq \mathcal{H}$, where $U \equiv H^\alpha(0,L)$, and $\alpha < 1$ is large enough so that the embedding of U into $C([0,L])$ is compact. We conclude by the Schauder fixed point theorem that there exists a fixed point of Φ in $C([0,T];U)$. This proves the existence part of the following theorem, which is one of the the main results in this work.

Theorem 4. *There exists a unique solution* (v,u,ζ) *to problem* (38)–(42) *and it satisfies:*

$$v \in \mathcal{V}, \quad u \in C([0,T];V), \quad v' \in \mathcal{V}',$$

$$\zeta' \in \mathcal{H},\ \zeta \in L^\infty\left(0,T;H^1(0,L)\right) \cap L^2\left(0,T;H^2(0,L)\right) \cap C([0,T];U).$$

For each $t \in [0,T]$

$$\zeta(x,t) \in [\delta,1] \quad a.e.\ x.$$

Proof. It only remains to verify the uniqueness of the solution. Suppose that (v_i, u_i, ζ_i), for $i = 1, 2$, are two solutions. We find from (38), using simple manipulations involving the relation between u and v, that

$$\frac{1}{2}\|v_1(t) - v_2(t)\|_H^2 + \frac{\delta}{2}\int_0^t ||v_1(s) - v_2(s)||_V^2\, ds$$

$$\leq K_\delta \int_0^t ||\zeta_1(s) - \zeta_2(s)||_{L^\infty(0,L)}^2 \left(||v_1(s)||_V^2 + 1\right) ds. \tag{46}$$

Now, using Lemma 2 again to the difference between the equations solved by ζ_i, we obtain

$$\frac{1}{2}\|\zeta_{1x}(t) - \zeta_{2x}(t)\|_H^2 + \frac{1}{2}\int_0^t \|\Delta(\zeta_1 - \zeta_2)\|_H^2 ds$$
$$\leq K(M)\int_0^t \left(\|\zeta_1 - \zeta_2\|_H^2 + \|u_{1x} - u_{2x}\|_H^2\right) ds.$$

Therefore, there is a positive constant C, independent of the solutions, such that

$$\|\zeta_{1x}(t) - \zeta_{2x}(t)\|_H^2 + \int_0^t \|\zeta_1 - \zeta_2\|_{H^2(0,L)}^2\, ds$$
$$\leq C\int_0^t \left(\|\zeta_1 - \zeta_2\|_H^2 + \int_0^s ||v_1 - v_2||_V^2\, dr\right) ds.$$

Similar, but somewhat simpler, computations using (41) yield

$$\frac{1}{2}\|\zeta_1(t) - \zeta_2(t)\|_H^2 + \frac{1}{2}\int_0^t \|\zeta_{1x} - \zeta_{2x}\|_H^2 ds$$
$$\leq C\int_0^t \left(\|\zeta_1 - \zeta_2\|_H^2 + \int_0^s ||v_1 - v_2||_V^2\, dr\right) ds.$$

Therefore,

$$||\zeta_1(t) - \zeta_2(t)||_V^2 + \int_0^t ||\zeta_1 - \zeta_2||_{H^2(0,L)}^2\, ds$$

$$\leq C\int_0^t \left(\|\zeta_1 - \zeta_2\|_H^2 + \int_0^s ||v_1 - v_2||_V^2\, dr\right) ds.$$

We use (46) to substitute into this inequality and obtain

$$||\zeta_1(t) - \zeta_2(t)||_V^2 + \int_0^t ||\zeta_1 - \zeta_2||_{H^2(0,L)}^2 \, ds$$

$$\leq C_\delta \int_0^t \Big(\|\zeta_1(s) - \zeta_2(s)\|_H^2$$

$$+ \int_0^s \Big(||\zeta_1(r) - \zeta_2(r)||_{L^\infty(0,L)}^2 \Big(||v_1(r)||_V^2 + 1 \Big) \Big) dr \Big) \, ds$$

We let $r < 2$ be large enough so that H^r embedds continuously into L^∞ and by the compactness of the embedding of H^2 into H^r, if $\varepsilon > 0$ we find

$$||\zeta_1(t) - \zeta_2(t)||_V^2 + \frac{1}{\varepsilon} \int_0^t ||\zeta_1 - \zeta_2||_{H^r(0,L)}^2 \, ds$$

$$\leq C_\varepsilon \int_0^t \|\zeta_1 - \zeta_2\|_H^2 ds + C_\delta \int_0^t \Big(\|\zeta_1(s) - \zeta_2(s)\|_H^2$$

$$+ \int_0^s \Big(||\zeta_1(r) - \zeta_2(r)||_{H^r(0,L)}^2 \Big(||v_1(r)||_V^2 + 1 \Big) \Big) dr \Big) \, ds.$$

Now, choosing ε small enough,

$$||\zeta_1(t) - \zeta_2(t)||_V^2 + \frac{1}{2\varepsilon} \int_0^t ||\zeta_1 - \zeta_2||_{H^r(0,L)}^2 \, ds$$

$$\leq C(\delta, \varepsilon) \int_0^t \int_0^s \Big(||\zeta_1(r) - \zeta_2(r)||_{H^r(0,L)}^2 \Big(||v_1(r)||_V^2 + 1 \Big) \Big) drds,$$

and by Gronwall's inequality $\zeta_1 = \zeta_2$, which implies by Lemma 1 that $v_1 = v_2$. This proves the theorem.

We note that the proof above implies the following corollary.

Corollary 1. *Consider problem* $(38) - (42)$*, then there exists* $T^* > 0$ *such that for* $t \in [0, T^*]$ *the function* ζ *in the solution provided in Theorem* 4 *stays within the interval* $(2\delta, 1]$ *so that every occurrence of* $\eta(\zeta)$ *in* $(38 - 42)$ *may be replaced with* ζ.

Proof. It follows from the fact that $\zeta \in C([0, T]; U)$, where U embeds continuously into $C([0, L])$, and ζ_0.

5 Estimates on strain

In this section we remove the truncation Q_M. Since the problem is one-dimensional, it suffices to obtain an estimate for u in $L^\infty\left(0,T;H^2(0,L)\right)$. We make additional assumptions on the problem data to obtain such an estimate, which involves pointwise bounds on u_x.

We assume the compatibility conditions on the initial data,

$$\begin{gathered}\Delta\zeta_0-\left(\zeta_0\mathcal{X}_{[l_1,l_2]}Q_M(u_{0x})-\varepsilon_0\right)_+\in H^1(0,L),\\ (cu_{0x})_x\in H.\end{gathered}\tag{47}$$

Let $\xi\equiv\zeta'$ and note that the time derivative of the source term in (41), $g(\zeta',v_x)$ is in $\mathcal{H}$. Therefore, there exists a unique solution to the problem

$$\xi'-\Delta\xi=g\left(\zeta',v_x\right),$$

$$\xi(0)=\Delta\zeta_0-\left(\zeta_0\mathcal{X}_{[l_1,l_2]}Q_M(u_{0x})-\varepsilon_0\right)_+\in H,$$

which satisfies $\xi'\in\mathcal{H},\ \xi\in L^2\left(0,T;H^2(0,L)\right)$. Then using Lemma 2, again, we obtain, for a.a. t,

$$\|\zeta'_x(t)\|_H^2=\|\xi_x(t)\|_H^2\le C(\zeta_0,\Delta\zeta_0,u_{0x}).\tag{48}$$

Similarly, an easier estimate for $\|\xi(t)\|_H^2$ is

$$\xi=\zeta'\in L^\infty\left(0,T;H^1(0,L)\right).\tag{49}$$

Also, as above, we obtain an estimate on $||\Delta\xi||_{L^2(0,T;H^2(0,L))}$ which yields the pointwise estimate

$$||\zeta||_{L^\infty(0,T;H^2(0,L))}\le C(\zeta_0,\Delta\zeta_0,u_{0x}),$$

which, in particular, implies that

$$||\zeta_x||_{L^\infty(0,T;L^\infty(0,L))}\le C(\zeta_0,\Delta\zeta_0,u_{0x}),\tag{50}$$

since in one dimension $H^1(0,L)$ embedds continuously into $L^\infty(0,L)$. One would need to work much harder if the problem were in a higher dimension.

We define the following time dependent family of functionals on H, which are convex, proper, and lower semicontinuous,

$$\phi(t,u)\equiv\begin{cases}\dfrac{1}{2}\displaystyle\int_0^L c(x)\,\zeta(x,t)\,u_x^2(x)\,dx & \text{if } u\in V,\\ +\infty \text{ if } u\notin V\end{cases}$$

Here, $t \in [0, T]$, and $D(\phi(t, \cdot)) = V$ is independent of t because $\delta \leq \zeta \leq 1$. Also, for $u \in V$,

$$\|\phi(t, u) - \phi(s, u)\|_H \leq \frac{1}{2} \int_0^L c(x) |\zeta(x, t) - \zeta(s, x)| u_x^2(x) \, dx$$

$$\begin{aligned} &\leq \frac{b}{2} ||\zeta(t) - \zeta(s)||_{L^\infty(0,L)} \int_0^L u_x^2(x) \, dx \\ &\leq \frac{b}{2} ||\zeta(t) - \zeta(s)||_{L^\infty(0,L)} \phi(r, u) \end{aligned}$$

$$\leq C\phi(r, u) \int_s^t \left|\left|\zeta'(\tau)\right|\right|_V d\tau \leq C\phi(r, u) |t - s|, \tag{51}$$

where $r \in [0, T]$ is arbitrary and we used (49). Also, the subgradient of $\phi(t, \cdot)$ is given by $\partial\phi(t, \cdot) = -(c(\cdot)\zeta(\cdot, t) u_x)_x$, and its domain is

$$\{u \in V : (c(\cdot)\zeta(\cdot, t) u_x)_x \in H\}.$$

Now consider (38)–(40) in which ζ is the solution satisfying (49), thanks to the compatibility condition (47) made on ζ_0. We have the following.

Lemma 3. *Assume that* (47) *holds and* $v_0 \in V$. *Then the solution to* (38)–(40) *satisfies* $v' \in \mathcal{H}$ *and* $(\zeta v_x)_x \in \mathcal{H}$.

Proof. Problem (38)–(40) is just an abstract form of the initial boundary value problem

$$\begin{aligned} v_t - (c\zeta v_x)_x - (c\zeta u_x)_x &= f, && (52) \\ v(0, t) &= v(L, t) = 0, && (53) \\ v(0) &= v_0, && (54) \\ u(t) &= u_0 + \int_0^t v(s) \, ds. && (55) \end{aligned}$$

The partial differential equation is of the form

$$v_t - \zeta_x(cv_x) - \zeta(cv_x)_x - \zeta_x(cu_x) - \zeta(cu_x)_x = f,$$

and when $(\zeta c v_x)_x \in \mathcal{H}$, it follows from the regularity of ζ, established earlier, that $(cv_x)_x \in \mathcal{H}$. Let $W \equiv \{v \in V : (cv_x)_x \in H\}$ and

$$\mathcal{W} \equiv \{v \in \mathcal{V} : (cv_x)_x \in \mathcal{H}\}$$

with the norm $||v||_{\mathcal{W}} \equiv ||(cv_x)_x||_{\mathcal{H}}$.

Let $v_1 \in \mathcal{W}$ and define $u_1(t) \equiv u_0 + \int_0^t v_1(s)\,ds$. Then, it follows from the main existence theorem in [14] that, given such ζ, there exists a unique solution v to the problem

$$\begin{aligned} & v_t - (c\zeta v_x)_x - \zeta_x c u_{1x} - \zeta (cu_{1x})_x \\ & \quad = v_t - \zeta_x (cv_x) - \zeta (cv_x)_x - \zeta_x c u_{1x} - \zeta (cu_{1x})_x = f, \\ & \quad v(0) = v_0, \end{aligned} \tag{56}$$

which satisfies $v_t \in \mathcal{H}$ and $(\zeta c v_x)_x \in \mathcal{H}$. Denote this v by $\Phi(v_1)$. Then consider $v_1, v_2 \in \mathcal{W}$ with the corresponding u_1, u_2. A similar argument as in Lemma 2 implies that we can multiply both sides of (56) by

$$-\left(c\Phi(v_1)_x\right)_x - \left(-\left(c\Phi(v_2)_x\right)_x\right)$$

and integrate by parts, eventually obtaining the estimate

$$\frac{1}{2}\|\sqrt{c}\left(\Phi(v_1)_x(t) - \Phi(v_2)_x(t)\right)\|_H^2 + \frac{\delta}{2}\int_0^t \|\left(c\Phi(v_1)_x\right)_x - \left(c\Phi(v_2)_x\right)_x\|_H^2 ds$$

$$\leq C\int_0^t \|c\Phi(v_1)_x(s) - c\Phi(v_2)_x(s)\|_H^2 ds$$

$$+C\int_0^t \|cu_{1x}(s) - cu_{2x}(s)\|_H^2 + \|\left(cu_{1x}\right)_x(s) - \left(cu_{2x}\right)_x(s)\|_H^2 ds$$

where here and below $C = C(\delta, \zeta_0, \Delta\zeta_0, u_{0x})$, and we used the fact $\zeta \geq \delta$ and the pointwise bound on ζ_x which follows from (50). After adjusting the constants, this simplifies to

$$\|\Phi(v_1)_x(t) - \Phi(v_2)_x(t)\|_H^2 + \int_0^t \|\left(c\Phi(v_1)_x\right)_x - \left(c\Phi(v_2)_x\right)_x\|_H^2 ds$$

$$\begin{aligned} \leq \;\; & C\int_0^t \|c\Phi(v_1)_x(s) - c\Phi(v_2)_x(s)\|_H^2 ds \\ & +C\int_0^t \int_0^s \|\left(cv_{1x}\right)_x - \left(cv_{2x}\right)_x\|_H^2 dr\,ds. \end{aligned}$$

Then,

$$\|\Phi(v_1)_x(t) - \Phi(v_2)_x(t)\|_H^2$$

$$\leq C\left[\int_0^t \|c\Phi\left(v_1\right)_x(s) - c\Phi\left(v_2\right)_x(s)\|_H^2 ds + \int_0^t\int_0^s ||v_1 - v_2||_W^2\, drds\right],$$

and by Gronwall's inequality and adjusting the constants, we obtain

$$\|\Phi\left(v_1\right)_x(t) - \Phi\left(v_2\right)_x(t)\|_H^2 \leq C\int_0^t\int_0^s ||v_1 - v_2||_W^2\, drds.$$

Now, integration over t yields

$$\int_0^t ||\Phi\left(v_1\right) - \Phi\left(v_2\right)||_W^2\, ds \leq C\int_0^t\int_0^s ||v_1 - v_2||_W^2\, drds,$$

where, as above $C = C\left(\delta, \zeta_0, \Delta\zeta_0, u_{0x}\right)$.

This estimate shows that a high enough power of Φ is a contraction mapping on $\mathcal{W}$, so there exists a unique fixed point v for Φ. This v is then the unique solution to (52) - (55). However, by the uniqueness of the weak solution to (38)-(40), it follows that v is the solution to the weak abstract problem. Also, we note that the construction yields

$$cv_x \in L^\infty\left(0, T; H\right).$$

This proves the lemma.

Now, since $\left(c\zeta v_x\right)_x \in \mathcal{H}$, it follows that $c\zeta v_x \in L^2\left(0, T; H^1\left(0, L\right)\right)$ and so $c\zeta v_x \in L^2\left(0, T; C\left([0, L]\right)\right)$, therefore

$$v_x \in L^2\left(0, T; L^\infty\left(0, L\right)\right),$$

thus $u_x \in C\left([0, T]; L^\infty\left(0, L\right)\right)$, hence,

$$u_x \in C\left([0, T]; L^\infty\left(0, L\right)\right).$$

This is the desired estimate on the strain which allows the elimination of the truncation function Q_M, proving the following local existence theorem.

Theorem 5. *Assume that the compatibility condition* (47) *holds,* $u_{0x}(x) < M$ *on* $[0, L]$*, where* M *is the truncation constant of* Q_M (43)*, and* $\zeta_0(x) \in (3\delta, 1]$*. Then, there exists* $T^* > 0$ *such that, for* $t \in [0, T^*)$*, the unique solution* (v, u, ζ) *of* $(38) - (42)$ *satisfies* $\eta\left(\zeta(t)\right) = \zeta(t)$ *and*

$$Q_M\left(u_x(t)\right)\mathcal{X}_{[l_1, l_2]}(x) = u_x^2(t)\,\mathcal{X}_{[l_1, l_2]}(x).$$

In addition, this solution has the following regularity,

$$c\zeta v_x \in L^2\left(0, T; H^1\left(0, L\right)\right), \quad v' \in \mathcal{H},$$

$$\zeta \in C\left([0, T]; H^2\left(0, L\right)\right),\ \zeta' \in L^2\left(0, T; H^2\left(0, L\right)\right).$$

6 Conclusions

Two models for the dynamic adhesive contact between two rods were presented. The first model assumes that the adhesive may be described as a rod made of an elastic-plastic material and then complete debonding occurs when the stress reaches the plasticity yield limit. In the second model the adhesive is also assumed to be a rod and the degradation of the adhesive is described by the introduction of material damage. Failure occurs when the material is completely damaged, or the damage reaches a critical floor value.

The analysis of the first model is routine. The second model was shown, in Section 4, to have a unique local (in time) weak solution. The proof was based on truncation of the strain energy and the damage function in the equation of motion. These allowed the use of standard tools to establish the existence of a weak global solution. Then, it was shown in Section 5 that with the appropriate initial conditions the weak solution is sufficiently regular so that the constraints (the truncations) are inactive on a time interval $[0, T^*)$, which means that the solution of the truncated problem is also the solution to the original problem.

Two quasistatic versions of the problem with material damage, with displacement or traction boundary condition at $x = 0$, were investigated in Section 3. The fact that the problems are one-dimensional allowed us to obtain a new condition for the damage source function, leading to the same and unusual parabolic nonlinear and nonlocal problem for the damage ζ, $P_{quas-\zeta}$ or $P_{\zeta q}$. The analysis of this problem will be done elsewhere.

In the limit when the thickness of the adhesive rod tends to zero a new adhesion source function was obtained, see the right-hand side of (26), which is unusual in that it contains ζ^{-1} which makes it non-Lipschitz, and different from the source functions used in [1, 2, 10, 20, 21, 22]. The problem was analyzed in Section 4.

Some future work, related issues, and unresolved questions follow. First, it may be of considerable interest to verify the model by comparing its predictions with experimental results. In this manner the model parameters may be estimated and then it may be used to predict the evolution of real systems. Because of the relative simplicity of the problem, it may be used as a bench-mark in applications, too.

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CONTROL OF DETERMINISTIC AND STOCHASTIC SYSTEMS WITH SEVERAL SMALL PARAMETERS – A SURVEY*

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Abstract

The past three decades of research on multiparameter singularly perturbed systems are reviewed, including recent results. Particular attention is paid to stability analysis, control, filtering problems and dynamic games. First, a parameter-independent design methodology is summarized, which employs a two-time-scale and descriptor systems approach without information on the small parameters. Further, various computational algorithms are included to avoid ill-conditioned systems: the exact slow-fast decomposition method, the recursive algorithm and Newton's method are considered in particular. Convergence results are presented and the existence and uniqueness of the solutions are discussed. Second, the new results obtained via the stochastic approach are presented. Finally, the results of a simulation of a practical power system are presented to validate the efficiency of the considered design methods.

keywords: Singular perturbations, several small parameters, deterministic systems, stochastic systems, robust control, Nash games.

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1 Introduction

When several small singular perturbation parameters of the same order of magnitude are present in the dynamic model of a physical system, the control problem is usually solved as a single parameter perturbation problem [18, 19, 21]; such a system is called a singularly perturbed system (SPS). Although this is achieved by scaling the coefficients, these parameters are often not known exactly. Thus, it is not applicable to a wider class of problems. One solution is to use the so-called multimodeling systems approach (see e.g. [1, 2, 7, 21, 22]). In addition, a joint multitime scale-multiparameter singularly perturbed system (MSPS) has been formulated [14, 23]. It should be noted that these small parameters are of different orders of magnitude.

Stability analysis, control and filtering problems in MSPSs have been thoroughly investigated. Multiarea power systems [1, 7] and passenger cars [15, 17, 29] can be modelled as MSPSs, which are widely used to represent system dynamics.

Since the investigations into the stability for the multimodel situation in [3, 4, 6], much of the interest in linear quadratic (LQ) control has been motivated by applications of the theory to multimodeling systems [1, 2, 12]. These interests in extending LQ control to dynamic games [5, 8, 9, 10, 13] were revealed. An overview of multimodeling control may be found in [11]. The recent theoretical advances in multimodeling techniques allow a revisiting of LQ control [49, 50, 52], the filtering problem [51, 54], the H_∞ control problem [48, 59], guaranteed cost control [56] and Nash games [53, 55, 57, 58]. A direct approach to the Lur'e problem for MSPSs has been proposed [27]. To extend the validity of continuous MSPSs, stability analysis, composite state feedback control and Nash games have been considered for discrete MSPSs [24, 25, 26].

In this paper, we present a survey of MSPSs in various control problems. Although many of the references consider deterministic problems, stochastic cases are also reviewed here. First, the results of stability analysis and the important related tests are given. After introducing the feature of the multiparameter algebraic Riccati equations (MARE) that is based on the LQ control for MSPSs, we discuss the two-time-scale design method for cases where the singular perturbation parameters are sufficiently small or unknown. However, iterative methods for finding the desired solutions are discussed when such parameters are known. In particular, to avoid ill-conditioned systems, the exact slow-fast decomposition method, recursive computation and Newton's method are surveyed. It is shown that these results are also valid for the filtering problem, H_∞ control problem, guaranteed cost control and Nash games. Moreover, some new results for stochastic systems that are governed by Itô differential equations are also discussed. Finally, it is shown that the concepts and methods surveyed in this paper can be exploited to solve the stochastic H_∞ control problem for an actual MSPS.

Notation: The notations used in this paper are fairly standard. **block diag** denotes the block diagonal matrix. $\det M$ denotes the determinant of M. $\mathrm{vec}M$ denotes an ordered stack of the columns of M. $\otimes$ denotes Kronecker product. $\mathrm{Re}\lambda(M)$ denotes a real part of $\lambda \in \mathbf{C}$ of M. $E[\cdot]$ denotes the expectation operator. The space of the $\Re^k$-valued functions that are quadratically integrable on $(0,\ \infty)$ are denoted by $L_2^k(0,\ \infty)$.

2 Stability

A general frame-work for the stability of a MSPS is formulated in [1, 3, 4, 6, 7, 21, 22]. Stability is very important for a linear or nonlinear MSPS when capturing the behaviour of the closed-loop MSPS. For a linear MSPS, the sufficient conditions for uniform asymptotic stability have been derived, and the asymptotic behaviour of the solution has also been investigated by using the transformation [1] and the D-stability [3]. In contrast, it is known that the Lyapunov method can be used to estimate the stability of a system by using a Lyapunov function without solving the nonlinear differential equations [4, 6]. The purpose of this section is to review the asymptotic stability for several sufficiently small parameters. These results are based on the asymptotic stability of a reduced-order slow system and fast subsystems.

A linear system of strongly coupled slow subsystem and weakly coupled fast subsystems is considered by (1).

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{N} A_{0j} z_j(t),\ x(0) = x^0, \tag{1a}$$

$$\varepsilon_i \dot{z}_i(t) = A_{i0} x(t) + A_{ii} z_i(t) + \sum_{j=1,\ j\neq i}^{N} \varepsilon_{ij} A_{ij} z_j(t),\ z_i(0) = z_i^0, \tag{1b}$$

where $x(t) \in \Re^{n_0}$ means the slow state vector. $z_i(t) \in \Re^{n_i}$, $i = 1, \dots, N$ mean the fast state vectors. All matrices above are of appropriate dimensions. The small singular perturbation parameters $\varepsilon_i > 0$, one per subsystem, represent time constant, inertias, masses etc., while the small regular perturbation parameters ε_{ij}, $i \neq j$ represent weak coupling between the subsystems.

The following result is well known for the stability of linear MSPS.

Lemma 1. *[1] If* $\mathrm{Re}\lambda(A_{ii}) < 0$, $i = 1, \dots, N$ *and* $\mathrm{Re}\lambda(A_s) < 0$, *then there exists a positive scalar* σ_1 *such that*

$$x(t) = x_s(t) + O(\|\varepsilon\|), \tag{2a}$$

$$z_i(t) = -A_{ii}^{-1} A_{i0} x_s(t) + z_{if}\left(\frac{t}{\varepsilon_i}\right) + O(\|\varepsilon\|), \tag{2b}$$

hold for all $t \in [0, \infty)$ *and all* $\varepsilon \in H$, $0 < \|\varepsilon\| \le \sigma_1$, *where*

$$\varepsilon := \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_N & \varepsilon_{12} & \cdots & \varepsilon_{N(N-1)} \end{bmatrix} \in \Re^{N^2},$$

$$H := \left\{ \varepsilon \in \Re^{N^2} \,\middle|\, m_{ij} \le \frac{\varepsilon_j}{\varepsilon_i} \le M_{ij},\ \bar{m}_{ij} \le \frac{\varepsilon_{ij}}{\varepsilon_i} \le \bar{M}_{ij},\right.$$
$$\left. m_{ij} > 0,\ \bar{m}_{ij} > 0,\ M_{ij} < \infty,\ \bar{M}_{ij} < \infty \right\},$$

$$\dot{x}_s(t) := A_s x_s(t), A_s := A_0 - \sum_{j=1}^{N} A_{0j} A_{jj}^{-1} A_{j0}, \dot{z}_{if}(t) := A_{ii} z_{if}(t), i = 1, \dots, N.$$

As an important implication, the following result is given for the stability of an uncertain MSPS.

Lemma 2. *[52] Let us consider uncertain MSPS*

$$\dot{x}(t) = [F_0 + O(\|\varepsilon\|)]x(t) + [F_{0f} + O(\|\varepsilon\|)]z(t),\ x(0) = x^0, \tag{3a}$$

$$\Pi_\varepsilon \dot{z}(t) = [F_{f0} + O(\|\varepsilon\|)]x(t) + [F_f + O(\|\varepsilon\|)]z(t),\ z(0) = z^0, \tag{3b}$$

where

$$\Pi_\varepsilon := \textbf{block diag}\begin{pmatrix} \varepsilon_1 I_{n_1} & \cdots & \varepsilon_N I_{n_N} \end{pmatrix},\ z(t) := \begin{bmatrix} z_1^T(t) & \cdots & z_N^T(t) \end{bmatrix}^T,$$

$$F_{0f} := \begin{bmatrix} F_{01} \cdots F_{0N} \end{bmatrix}, F_{f0} := \begin{bmatrix} F_{10}^T \cdots F_{N0}^T \end{bmatrix}^T, F_f := \textbf{block diag}\begin{pmatrix} F_{11} \cdots F_{NN} \end{pmatrix},$$

$x(t) \in \Re^{n_0}$ and $z_i(t) \in \Re^{n_i}$, $i = 1, \ldots, N$ are the state vectors. All matrices above are of appropriate dimensions.

If F_{ii}, $i = 1, \ldots, N$ and $\bar{F} = F_0 - \sum_{j=1}^{N} F_{0j} F_{jj}^{-1} F_{j0}$ are stable, then there exists a positive scalar σ_2 such that for all $t \in [0, \infty)$ and all $\varepsilon \in H$, $0 < \|\varepsilon\| \leq \sigma_2$, uncertain MSPS (3) *is asymptotically stable.*

Asymptotic expansions of the solutions as well as the problem of exponential stability of the zero state equilibrium of a singularly perturbed linear system with several small parameters of different orders of magnitude may be found in [39], see also Chapter 3 in [40].

At the end of this section, sufficient conditions are stated to guarantee the asymptotic stability of a class of nonlinear SPS with several perturbation parameters of the same order. Now, let us consider the nonlinear MSPS given by (4).

$$\dot{x}(t) = f(t, x) + F(t, x)z(t), \tag{4a}$$

$$\Pi_\varepsilon \dot{z}(t) = g(t, x) + G(t, x)z(t). \tag{4b}$$

We assume that the following conditions are satisfied for all $x(t) \in S_x$, where S_x is a closed set in $\Re^{n_0}$ containing the origin and for all $t \geq t_0$.

(a) $x(t) = 0$ is the unique point in S_x for which $f(t, 0) = 0$ and $g(t, 0) = 0$.

(b) f, g, F, G and $h := G^{-1}(t, x)g(t, x)$ are bounded and satisfy the necessary smoothness requirements for existence, uniqueness and continuity of the solution of (4). Moreover, $G(t, x)$ and $h(t, x)$ have bounded first partial derivatives with respect to t and $x(t)$.

(c) There exists a positive definite Lyapunov function $V(t, x)$ such that

$$V_t + V_x f_0(t, x) \leq -\kappa_1 \psi^2(x),\ \|V_x F(t, x)\| \leq \kappa_2 \psi(x),$$
$$\|h_t + h_x f_0(t, x)\| \leq \kappa_3 \psi(x),$$
$$f_0(t, x) := f(t, x) - F(t, x)h(t, x),\ V_t := \frac{\partial V}{\partial t},\ V_x := \frac{\partial V}{\partial x},$$
$$h_t := \frac{\partial h}{\partial t},\ h_x := \frac{\partial h}{\partial x},$$

where $\psi(x)$ is a positive definite function of $x(t)$, κ_1, κ_2 and κ_3 are positive scalars.

(d) The real parts of the eigenvalues of $\Pi_\varepsilon^{-1}G$ are strictly negative, that is $\mathrm{Re}\lambda(\Pi_\varepsilon^{-1}G) \leq -\tau < 0$ for all $\varepsilon \in H$, where τ is a positive scalar independent of t, x and ε.

The asymptotic stability of equation (4) is established in the following basic lemma.

Lemma 3. *[6] Under conditions* (a)-(d)*, there exists a positive scalar σ_3 such that for all $\varepsilon \in H$ with $0 < \|\varepsilon\| \leq \sigma_3$, the origin $x = 0$, $z = 0$ is an asymptotically stable equilibrium point of* (4)*.*

It should be observed that in practice, Lemma 1 is included in Lemma 3 as a special case.

For the problem of exponential stability of a singularly perturbed linear system with state delays we refer to [16] and [41].

3 Linear Quadratic Regulator (LQR) Problem

The solution of a LQ regulator (LQR) problem is usually given in the form of state feedback control. Indeed, the LQR technique was used to solve the active suspension control problem [29]. In this section, we discuss the LQR problems from the point of view of the reduced-order technique and numerical aspects. These results will be covered as the extension of SPS [18, 19, 21].

3.1 Two-Time-Scale Decomposition

When the small perturbation parameters ε_i are not known, a popular approach to deal with the MSPS is the two-time-scale decomposition method (see e.g. [1, 21]). In practice, since ε_i is very small or unknown, the previous technique is very efficient. First, the LQ control problem for the MSPS was studied by using composite controller design [1, 2]. In [2], the resulting near-optimal controller has been proven to have a performance level, i.e. $O(\|\varepsilon\|)$, where $\|\varepsilon\|$ denotes the norm of the vector $\varepsilon := [\varepsilon_1 \ \cdots \ \varepsilon_N]$, close to the optimal performance level for the standard and nonstandard MSPS. However, one major drawback of this method is that the fast state matrices A_{ii} are invertible. Indeed, if this condition holds, we cannot obtain the reduced-order slow subsystems. To avoid the invertibility assumptions, the descriptor systems approach [28] can be used. The descriptor systems approach will be discussed later as a nonstandard MSPS. Although the descriptor systems approach can still be used for general MSPSs, the two-time-scale decomposition method is recommended in this case because the fast state matrices are invertible in most practical systems. Some properties of the two-time-scale decomposition method are described next.

We consider a specific structure of N-lower level multi-fast subsystems interconnected through the dynamics of a higher level slow subsystem [1, 7, 52].

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{j=1}^{N} A_{0j} z_j(t) + \sum_{j=1}^{N} B_{0j} u_j(t),\ x(0) = x^0, && \text{(5a)} \\ \varepsilon_i \dot{z}_i(t) &= A_{i0} x(t) + A_{ii} z_i(t) + B_{ii} u_i(t),\ z_i(0) = z_i^0,\ i = 1,\ \dots,N, && \text{(5b)} \end{aligned}$$

where $u_i(t) \in \Re^{m_i}$, $i = 1,\ \dots,N$ are the control inputs.

It should be noted that all fast state matrices A_{ii}, $i = 1,\ \dots,N$ are invertible. The performance criterion is given by

$$J = \frac{1}{2} \int_0^{\infty} \left(\xi^T(t) Q \xi(t) + \sum_{j=1}^{N} u_j^T(t) R_j u_j(t) \right) dt, \tag{6}$$

where

$$\begin{aligned} \xi(t) &:= \begin{bmatrix} x^T(t) & z_1^T(t) & \cdots & z_N^T(t) \end{bmatrix}^T \in \boldsymbol{R}^{\bar{n}},\ \bar{n} := \sum_{j=0}^{N} n_j, \\ Q &:= C^T C = \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^T & Q_f \end{bmatrix},\ Q_{00} \ := \ C_0^T C_0 = \sum_{j=0}^{N} C_{j0}^T C_{j0}, \\ Q_{0f} &:= C_0^T C_f = \begin{bmatrix} Q_{01} & \cdots & Q_{0N} \end{bmatrix} = \begin{bmatrix} C_{10}^T C_{11} & \cdots & C_{N0}^T C_{NN} \end{bmatrix}, \\ Q_f &:= C_f^T C_f = \textbf{block diag} \begin{pmatrix} Q_{11} & \cdots & Q_{NN} \end{pmatrix} = \\ & \quad \textbf{block diag} \begin{pmatrix} C_{11}^T C_{11} & \cdots & C_{NN}^T C_{NN} \end{pmatrix}, \end{aligned}$$

$$
\begin{aligned}
C &:= \begin{bmatrix} C_0 & C_f \end{bmatrix},\ C_0 := \begin{bmatrix} C_{00} \\ C_{10} \\ \vdots \\ C_{N0} \end{bmatrix}, \\
C_f &:= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_{11} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_{NN} \end{bmatrix}, \\
R &:= \textbf{block diag}\begin{pmatrix} R_1 & \cdots & R_N \end{pmatrix}.
\end{aligned}
$$

Let the optimal control for the LQ control problem (5) and (6) be

$$u_{\mathrm{opt}}(t) = K_{\mathrm{opt}}\xi(t) = -R^{-1}B_\varepsilon^T P_\varepsilon \xi(t), \tag{7}$$

where P_ε satisfies the MARE

$$P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon - P_\varepsilon S_\varepsilon P_\varepsilon + Q = 0, \tag{8}$$

with

$$
\begin{aligned}
A_\varepsilon &:= \begin{bmatrix} A_0 & A_{0f} \\ \Pi_\varepsilon^{-1}A_{f0} & \Pi_\varepsilon^{-1}A_f \end{bmatrix}, \\
A_{0f} &:= \begin{bmatrix} A_{01} & \cdots & A_{0N} \end{bmatrix},\ A_{f0} := \begin{bmatrix} A_{10}^T & \cdots & A_{N0}^T \end{bmatrix}^T, \\
A_f &:= \textbf{block diag}\begin{pmatrix} A_{11} & \cdots & A_{NN} \end{pmatrix}, \\
S_\varepsilon &:= B_\varepsilon R^{-1} B_\varepsilon^T = \begin{bmatrix} S_{00} & S_{0f}\Pi_\varepsilon^{-1} \\ \Pi_\varepsilon^{-1}S_{0f}^T & \Pi_\varepsilon^{-1}S_f\Pi_\varepsilon^{-1} \end{bmatrix}, \\
S_{00} &:= B_0 R^{-1} B_0^T = \sum_{j=1}^{N} B_{0j}R_j^{-1}B_{0j}^T, \\
S_{0f} &:= B_0 R^{-1} B_f^T = \begin{bmatrix} S_{01} & \cdots & S_{0N} \end{bmatrix} = \\
&\quad \begin{bmatrix} B_{01}R_1^{-1}B_{11}^T \cdots & B_{0N}R_N^{-1}B_{NN}^T \end{bmatrix}, \\
S_f &:= B_f R^{-1} B_f^T = \textbf{block diag}\begin{pmatrix} S_{11} & \cdots & S_{NN} \end{pmatrix} = \\
&\quad \textbf{block diag}\begin{pmatrix} B_{11}R_1^{-1}B_{11}^T & \cdots & B_{NN}R_N^{-1}B_{NN}^T \end{pmatrix}, \\
B_\varepsilon &:= \begin{bmatrix} B_0 \\ \Pi_\varepsilon^{-1}B_f \end{bmatrix},\ B_0 := \begin{bmatrix} B_{01} & \cdots & B_{0N} \end{bmatrix}, \\
B_f &:= \textbf{block diag}\begin{pmatrix} B_{11} & \cdots & B_{NN} \end{pmatrix}.
\end{aligned}
$$

However, we cannot solve the MARE (8) without the knowledge of the small perturbation parameters ε_i. When ε_i is very small or unknown, the two-time-scale design method [1, 52] is very efficient.

According to [1, 7], the near-optimal closed-loop control is given by

$$\begin{aligned} u_{i\mathrm{com}}(t) &= -[(I_{m_i} - R_i^{-1} B_{ii}^T X_{ii} A_{ii}^{-1} B_{ii}) \tilde{R}_i^{-1} (\tilde{D}_i^T \tilde{C}_{i0} + \tilde{B}_{0i}^T X_{00}) \\ &\quad + R_i^{-1} B_{ii}^T X_{ii} A_{ii}^{-1} A_{i0}] x(t) - R_i^{-1} B_{ii}^T X_{ii} z_i(t),\ i = 1,\ \dots, N, \end{aligned} \tag{9}$$

where $\tilde{B}_{0i} = B_{0i} - A_{0i} A_{ii}^{-1} B_{ii}$, $\tilde{C}_{i0} = C_{i0} - C_{ii} A_{ii}^{-1} A_{i0}$, $\tilde{R}_i = R_i + \tilde{D}_i^T \tilde{D}_i$, $\tilde{D}_i = -C_{ii} A_{ii}^{-1} B_{ii}$.

In the above, X_{00} is the unique stabilizing positive semidefinite symmetric solution of the following algebraic Riccati equation (ARE)

$$\begin{aligned} &X_{00}(A_s - B_s R_s^{-1} D_s^T C_s) + (A_s - B_s R_s^{-1} D_s^T C_s)^T X_{00} - \\ &\quad - X_{00} B_s R_s^{-1} B_s^T X_{00} + C_s^T (I_{\bar{l}} - D_s R_s^{-1} D_s^T) C_s = 0, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &R_s = R + D_s^T D_s,\ B_s = B_0 - A_{0f} A_f^{-1} B_f = \\ &\begin{bmatrix} B_{01} - A_{01} A_{11}^{-1} B_{11} & \cdots & B_{0N} - A_{0N} A_{NN}^{-1} B_{NN} \end{bmatrix}, \\ &C_s = C_0 - C_f A_f^{-1} A_{f0} = \big[\, C_{00}^T \quad (C_{10} - C_{11} A_{11}^{-1} A_{10})^T \quad \cdots \\ &\qquad\qquad\qquad\qquad (C_{N0} - C_{NN} A_{NN}^{-1} A_{N0})^T \big]^T, \\ &D_s = -C_f A_f^{-1} B_f = -\begin{bmatrix} 0 & \cdots & 0 \\ C_{11} A_{11}^{-1} B_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{NN} A_{NN}^{-1} B_{NN} \end{bmatrix}. \end{aligned}$$

$X_{ii},\ i = 1,\ \dots, N$ are the unique stabilizing positive semidefinite solution of the following AREs

$$X_{ii} A_{ii} + A_{ii}^T X_{ii} - X_{ii} S_{ii} X_{ii} + Q_{ii} = 0. \tag{11}$$

It is well known from [1] that the controller (9) is identical with the following controller

$$u_{i\mathrm{com}}(t) = -R_i^{-1} B_{i0}^T X_{00} x(t) - R_i^{-1} B_{ii}^T X_{i0} x(t) - R_i^{-1} B_{ii}^T X_{ii} z_i(t), \tag{12}$$

where $X_{i0},\ i = 1,\ \dots, N$ are

$$X_{i0}^T = [X_{00}(S_{0i} X_{ii} - A_{0i}) - (A_{i0}^T X_{ii} + Q_{0i})](A_{ii} - S_{ii} X_{ii})^{-1}. \tag{13}$$

Furthermore, the composite controller $u_{\text{com}}(t) = \left[\begin{array}{ccc} u_{1\text{com}}(t)^T & \cdots & u_{N\text{com}}(t)^T \end{array} \right]^T$ can be rewritten as the following composite controller

$$u_{\text{com}}(t) := K_{\text{com}}\xi(t) = -R^{-1}B^T \begin{bmatrix} X_{00} & 0 & 0 & \cdots & 0 \\ X_{10} & X_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{N0} & 0 & 0 & \cdots & X_{NN} \end{bmatrix} \xi(t). \quad (14)$$

Theorem 1. *[1] There exists a positive scalar $\bar{\sigma}_1$ such that for all $\varepsilon \in H$ with $0 < \|\varepsilon\| \leq \bar{\sigma}_1$ the closed loop MSPS* (5) *is asymptotically stable. Furthermore, the use of the composite controller* (14) *results in J_{app} satisfying*

$$\lim_{\|\varepsilon\| \to +0} (J_{\text{com}} - J_{\text{opt}}) = 0, \quad (15)$$

where $J_{\text{opt}} = \xi^T(0)P_\varepsilon\xi(0)$ and $J_{\text{com}} = \xi^T(0)W_\varepsilon\xi(0)$ with

$$W_\varepsilon(A_\varepsilon + B_\varepsilon K_{\text{com}}) + (A_\varepsilon + B_\varepsilon K_{\text{com}})^T W_\varepsilon + K_{\text{com}}^T R K_{\text{com}} + Q = 0.$$

According to Theorem 1, the detailed cost degradation has not been established. This property is described in a subsequent section.

3.2 Matrix Riccati Equations

The multimodel strategies for the LQ control problem are given in terms of Riccati or Riccati-type equations, which are parameterized by several small positive perturbation parameters. The existence of a unique and bounded solution to the MARE (8) was first shown in [13]. This important result is summarized as follows.

Since the matrices A_ε and B_ε contain the term of ε_i^{-1}, a solution P_ε of the MARE (8), if it exists, must contain terms of ε_i. Taking this fact into consideration, we look for a solution P_ε of the MARE (8) with the structure

$$P_\varepsilon := \begin{bmatrix} P_{00} & P_{f0}^T \Pi_\varepsilon \\ \Pi_\varepsilon P_{f0} & \Pi_\varepsilon P_f \end{bmatrix}, \ P_{00} = P_{00}^T,$$

$$P_{f0} := \begin{bmatrix} P_{10} \\ \vdots \\ P_{N0} \end{bmatrix},$$

$$P_f := \begin{bmatrix} P_{11} & \alpha_{12}P_{21}^T & \alpha_{13}P_{31}^T & \cdots & \alpha_{1N}P_{N1}^T \\ P_{21} & P_{22} & \alpha_{23}P_{32}^T & \cdots & \alpha_{2N}P_{N2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{(N-1)1} & P_{(N-1)2} & P_{(N-1)3} & \cdots & \alpha_{(N-1)N}P_{N(N-1)}^T \\ P_{N1} & P_{N2} & P_{N3} & \cdots & P_{NN} \end{bmatrix},$$

$\Pi_\varepsilon P_f = P_f^T \Pi_\varepsilon$.

In order to guarantee the existence of the reduced-order ARE and its standard stabilizability and the detectability conditions when $\|\varepsilon\| \to +0$, Assumptions 1 and 2 are needed.

Assumption 1. *The triples* $(A_{ii},\ B_{ii},\ C_{ii})$, $i = 1,\ \dots\ , N$ *are stabilizable and detectable.*

Assumption 2.

$$\operatorname{rank} \begin{bmatrix} sI_{n_0} - A_0 & -A_{0f} & B_0 \\ -A_{f0} & -A_f & B_f \end{bmatrix} = \bar{n}, \tag{16a}$$

$$\operatorname{rank} \begin{bmatrix} sI_{n_0} - A_0^T & -A_{f0}^T & C_0^T \\ -A_{0f}^T & -A_f^T & C_f^T \end{bmatrix} = \bar{n}, \tag{16b}$$

with $\forall s \in \mathbf{C}$, $\mathrm{Re}[s] \geq 0$.

Before investigating the optimal control problem, we investigate the asymptotic structure of the MARE (8).

The MARE (8) can be partitioned into

$$\begin{aligned} f_1 &= P_{00}^T A_0 + A_0^T P_{00} + P_{f0}^T A_{f0} + A_{f0}^T P_{f0} - P_{00}^T S_{00} P_{00} - \\ &\quad - P_{f0}^T S_f P_{f0} - P_{00}^T S_{0f} P_{f0} - P_{f0}^T S_{0f}^T P_{00} + Q_{00} = 0, \quad (17a) \\ f_2 &= A_0^T P_{f0}^T \Pi_\varepsilon + A_{f0}^T P_f + P_{00}^T A_{0f} + P_{f0}^T A_f - P_{00}^T S_{00} P_{f0}^T \Pi_\varepsilon - \\ &\quad - P_{f0}^T S_{0f}^T P_{f0}^T \Pi_\varepsilon - P_{00}^T S_{0f} P_f - P_{f0}^T S_f P_f + Q_{0f} = 0, \quad (17b) \\ f_3 &= P_f^T A_f + A_f^T P_f + \Pi_\varepsilon P_{f0} A_{0f} + A_{0f}^T P_{f0}^T \Pi_\varepsilon - P_f^T S_f P_f \\ &\quad - P_f^T S_{0f}^T P_{f0}^T \Pi_\varepsilon - \Pi_\varepsilon P_{f0} S_{0f} P_f - \Pi_\varepsilon P_{f0} S_{00} P_{f0}^T \Pi_\varepsilon + Q_f = 0. \quad (17c) \end{aligned}$$

It is assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero (see e.g., [1, 2]), that is

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \alpha_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \frac{\varepsilon_j}{\varepsilon_i}. \tag{18}$$

Assumption 1 ensures that $A_{ii} - S_{ii}\bar{P}_{ii}^*$, $i = 1, \dots, N$ are nonsingular. Substituting the solution of (17c) into (17b) and substituting $\bar{P}_{f0}^*$ into (17a) and making some lengthy calculations (the detail is omitted for brevity), we obtain the following zeroth-order equations (19)

$$\bar{P}_{00}^* \boldsymbol{A} + \boldsymbol{A}^T \bar{P}_{00}^* - \bar{P}_{00}^* \boldsymbol{S} \bar{P}_{00}^* + \boldsymbol{Q} = 0, \tag{19a}$$
$$\bar{P}_{f0}^* = -N_2^T + N_1^T \bar{P}_{00}^*, \tag{19b}$$
$$\bar{P}_f^* A_f + A_f^T \bar{P}_f^* - \bar{P}_f^* S_f \bar{P}_f^* + Q_f = 0, \tag{19c}$$

where

$$\begin{aligned}
\boldsymbol{A} &:= A_0 + N_1 A_{f0} + S_{0f} N_2^T + N_1 S_f N_2^T, \\
\boldsymbol{S} &:= S_{00} + N_1 S_{0f}^T + S_{0f} N_1^T + N_1 S_f N_1^T, \\
\boldsymbol{Q} &:= Q_{00} - N_2 A_{f0} - A_{f0}^T N_2^T - N_2 S_f N_2^T, \\
\bar{P}_{f0}^* &:= \begin{bmatrix} \bar{P}_{10}^{*T} & \cdots & \bar{P}_{N0}^{*T} \end{bmatrix}^T, \ \bar{P}_f^* := \textbf{block diag}\begin{pmatrix} \bar{P}_{11}^* & \cdots & \bar{P}_{NN}^* \end{pmatrix}, \\
\bar{P}_{i0}^{*T} &:= -[\bar{P}_{00}^* D_{0i} + (A_{i0}^T \bar{P}_{ii}^* + Q_{0i})] D_{ii}^{-1}, \\
& \quad \bar{P}_{ii}^* A_{ii} + A_{ii}^T \bar{P}_{ii}^* - \bar{P}_{ii}^* S_{ii} \bar{P}_{ii}^* + Q_{ii} = 0, \ i = 1, \dots, N, \\
N_1^T &:= -\bar{A}_f^{-T} \bar{A}_{0f}^T = \begin{bmatrix} -D_{01} D_{11}^{-1} \cdots -D_{0N} D_{NN}^{-1} \end{bmatrix}^T = \begin{bmatrix} N_{11} \cdots N_{1N} \end{bmatrix}^T, \\
N_2^T &:= \bar{A}_f^{-T} \bar{Q}_{0f}^T = \begin{bmatrix} \bar{Q}_{01} D_{11} & \cdots & \bar{Q}_{0N} D_{NN} \end{bmatrix}^T = \begin{bmatrix} N_{21} & \cdots & N_{2N} \end{bmatrix}^T, \\
\bar{A}_{0f} &:= A_{0f} - S_{0f} \bar{P}_f^* = \begin{bmatrix} D_{01} & \cdots & D_{0N} \end{bmatrix}, \\
\bar{A}_f &:= A_f - S_f \bar{P}_f^* = \textbf{block diag}\begin{pmatrix} D_{11} & \cdots & D_{NN} \end{pmatrix}, \\
\bar{Q}_{0f} &:= Q_{0f} + A_{f0}^T \bar{P}_f^* = \begin{bmatrix} \bar{Q}_{01} & \cdots & \bar{Q}_{0N} \end{bmatrix}, \\
D_{0i} &:= A_{0i} - S_{0i} \bar{P}_{ii}^*, \ D_{ii} := A_{ii} - S_{ii} \bar{P}_{ii}^*, \\
\bar{Q}_{0i} &:= Q_{0i} + A_{i0}^T \bar{P}_{ii}^*, \ i = 1, \dots, N.
\end{aligned}$$

In the following we established the relation between the MARE (8) and the zeroth-order equations (19). Before doing that, we give the results for the AREs (19).

Lemma 4. *[52] Under Assumptions 1 and 2, the following results hold.*

(i) *The matrices* $\boldsymbol{A}$, $\boldsymbol{S}$ *and* $\boldsymbol{Q}$ *do not depend on* $\bar{P}_{ii}^*$, $i = 1, \ldots, N$. *That is, following formulations are satisfied.*

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{S} \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{bmatrix} = T_{00} - \sum_{j=1}^{N} T_{0j} T_{jj}^{-1} T_{j0}, \tag{20}$$

where

$$T_{00} := \begin{bmatrix} A_0 & -S_{00} \\ -Q_{00} & -A_0^T \end{bmatrix}, \ T_{0i} := \begin{bmatrix} A_{0i} & -S_{0i} \\ -Q_{0i} & -A_{i0}^T \end{bmatrix},$$
$$T_{i0} := \begin{bmatrix} A_{i0} & -S_{0i}^T \\ -Q_{0i}^T & -A_{0i}^T \end{bmatrix}, \ T_{ii} := \begin{bmatrix} A_{ii} & -S_{ii} \\ -Q_{ii} & -A_{ii}^T \end{bmatrix}, \ i = 1, \ldots, N.$$

(ii) *There exist a matrix* $\boldsymbol{B} := \begin{bmatrix} B_{01} + N_{11}B_{11} & \cdots & B_{0N} + N_{1N}B_{NN} \end{bmatrix} \in \boldsymbol{R}^{n_0 \times \bar{m}}$, $\bar{m} := \sum_{j=1}^{N} m_j$ *and a matrix* $\boldsymbol{C}$ *with the same dimension as* C_0 *such that* $\boldsymbol{S} = \boldsymbol{B}R^{-1}\boldsymbol{B}^T$, $\boldsymbol{Q} = \boldsymbol{C}^T\boldsymbol{C}$. *Moreover, the triple* $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ *is stabilizable and detectable.*

Remark 1. *Note the relation*

$$T_{ii} := \begin{bmatrix} A_{ii} & -S_{ii} \\ -Q_{ii} & -A_{ii}^T \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ \bar{P}_{ii}^* & I_{n_i} \end{bmatrix} \begin{bmatrix} D_{ii} & -S_{ii} \\ 0 & -D_{ii}^T \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ -\bar{P}_{ii}^* & I_{n_i} \end{bmatrix}.$$

Since T_{ii} *is nonsingular under Assumption 1 and the ARE* (19c) *has a stabilizing solution under Assumption 2,* D_{ii} *is also nonsingular. This means that* T_{ii}^{-1} *can be expressed explicitly in terms of* D_{ii}^{-1}. *Using the similar manner, we have the following relations.*

$$T_{ii}^{-1} = \begin{bmatrix} I_{n_i} & 0 \\ \bar{P}_{ii}^* & I_{n_i} \end{bmatrix} \begin{bmatrix} D_{ii}^{-1} & -D_{ii}^{-1}S_{ii}D_{ii}^{-T} \\ 0 & -D_{ii}^{-T} \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ -\bar{P}_{ii}^* & I_{n_i} \end{bmatrix}.$$

Theorem 2. *[13, 52] Under Assumptions 1 and 2, there exists a positive scalar* $\bar{\sigma}_2$ *such that for all* $\varepsilon \in H$ *with* $0 < \|\varepsilon\| \leq \bar{\sigma}_2$ *the MARE* (8) *admits a symmetric positive semidefinite stabilizing solution* P_ε *which can be written as*

$$P_\varepsilon = \Phi_\varepsilon \begin{bmatrix} \bar{P}_{00}^* + O(\|\varepsilon\|) & [\bar{P}_{f0}^* + O(\|\varepsilon\|)]^T \Pi_\varepsilon \\ \bar{P}_{f0}^* + O(\|\varepsilon\|) & \bar{P}_f^* + O(\|\varepsilon\|) \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} \bar{P}_{00}^* + O(\|\varepsilon\|) & [\bar{P}_{f0}^* + O(\|\varepsilon\|)]^T \Pi_\varepsilon \\ \Pi_\varepsilon [\bar{P}_{f0}^* + O(\|\varepsilon\|)] & \Pi_\varepsilon [\bar{P}_f^* + O(\|\varepsilon\|)] \end{bmatrix},$$

where $\Phi_\varepsilon = \textbf{block diag} \begin{pmatrix} I_{n_0} & \varepsilon_1 I_{n_1} & \cdots & \varepsilon_N I_{n_N} \end{pmatrix}$.

This result can be easily extended to the other multimodeling-type ARE (see e.g., [48, 51, 53]). The cross-coupled MARE is discussed later.

3.3 Nonstandard MSPS

If one of the fast state matrices A_{ii}, $j = 1, \ldots, N$ is singular, the MSPS is called a nonstandard MSPS. In such a case, we cannot utilize the two-time-scale decomposition technique.

Recent theoretical advances in the descriptor system approach allow a revisiting of the various control problems [28]. Since the feedback controller in such problems can be expressed by solutions of the reduced-order and parameter independent AREs, the resulting feedback is derived without invertibility assumptions.

We focus on a specific linear state feedback controller which does not depend on the values of the small parameters. Our methodology is different from the methodology of [1]. This design method is based on the descriptor system approach. If $\|\varepsilon\|$ is very small, it is obvious that the optimal linear state feedback controller (7) can be approximated as

$$u_{\text{app}}(t) = K_{\text{app}}\xi(t) = -R^{-1}B^T \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix} \xi(t), \tag{22}$$

where

$$\bar{P}_{i0}^* = \begin{bmatrix} \bar{P}_{ii}^* & -I_{n_i} \end{bmatrix} T_{ii}^{-1} T_{i0} \begin{bmatrix} I_{n_0} \\ \bar{P}_{00}^* \end{bmatrix}.$$

Theorem 3. *[52] Under Assumptions 1 and 2, the use of the approximation controller* (22) *results in* J_{app} *satisfying*

$$J_{\text{app}} = J_{\text{opt}} + O(\|\varepsilon\|^2), \tag{23}$$

where$J_{\text{app}} = \xi^T(0) U_\varepsilon \xi(0)$ *with*

$$U_\varepsilon(A_\varepsilon + B_\varepsilon K_{\text{app}}) + (A_\varepsilon + B_\varepsilon K_{\text{app}})^T U_\varepsilon + K_{\text{app}}^T R K_{\text{app}} + Q = 0.$$

The following theorem gives a relation between the composite controller (14) and the approximate controller (22).

Theorem 4. *[52] Under Assumptions 1 and 2, the following identities*

$$X_{ii} = \bar{P}_{ii}^*, \ X_{i0} = \bar{P}_{i0}^*, \ X_{00} = \bar{P}_{00}^*, \ i = 1, \ \dots, N \tag{24}$$

hold. Hence the resulting composite controller (14) *is the same as the composite optimal controller* (22).

It can be observed that the new near-optimal controller (22) is equivalent to the existing one [1] in the case of the standard and the nonstandard MSPSs. We claim that the proposed controller (22) includes the composite near-optimal controller [1] as a special case since the proposed controller can be constructed even if the fast state matrices are singular.

3.4 Numerical Algorithms

In order to obtain the optimal solution to the multimodeling problems, we must solve the MARE, which are parameterized by the small, positive parameters ε_i, $i = 1, \ \dots, N$, which have the same order of magnitude. Various reliable approaches to the theory of ARE have been well documented in many literatures (see e.g. [32, 33]). One of the approaches is the invariant subspace approach, which is based on the Hamiltonian matrix [32]. However, such an approach is not adequate for the MSPS since the workspace dimensions required to carry out the calculations for the Hamiltonian matrix are twice those of the original full-system. Another disadvantage is that there is no guarantee of symmetry for the solution of the ARE when the ARE is known to be ill-conditioned [32]. It should be noted that it is very difficult to solve the MARE due to the high dimension and numerical stiffness [18, 19]. To avoid this drawback, various reliable approaches for solving the MARE have been well documented. Three types of numerical algorithms are presented in this paper: the first one is the exact slow-fast decomposition method, the second is a recursive algorithm and the third one is Newton's method.

3.4.1 Exact Slow-fast Decomposition Method

The exact slow-fast decomposition method for solving the MARE has been tackled in [15]. In order to simplify the notation, $N = 2$ is summarized [15].

Let us consider the nonlinear matrix algebraic equations.

$$T_{11}L_1 - T_{10} - \varepsilon_1 L_1(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1) = 0, \tag{25a}$$
$$T_{22}L_2 - \alpha_{12}L_3T_{10} - T_{20} - \varepsilon_2 L_2(T_{00} - T_{02}L_2) = 0, \tag{25b}$$
$$T_{22}L_3 - \alpha_{12}L_3T_{11} - \varepsilon_2 L_2(T_{01} - T_{02}L_3) = 0, \tag{25c}$$
$$-H_1T_{11} - \varepsilon_1 H_1L_1(T_{01} - T_{02}L_3) + (T_{01} - T_{02}L_3) + \varepsilon_1(T_{00} - \tag{25d}$$
$$-T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)H_1 = 0,$$
$$-H_2T_{22} + \alpha_{12}T_{11}H_2 + \varepsilon_2 L_1(T_{01} - T_{02}L_3)H_2 + \tag{25e}$$
$$+(L_1 - \varepsilon_2 H_2L_2)T_{02} = 0,$$
$$-H_3T_{22} - \varepsilon_2 H_3L_2T_{02} - \varepsilon_2(T_{01} - T_{02}L_3) - T_{02} + \varepsilon_2(T_{00} - \tag{25f}$$
$$-T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)H_3 = 0.$$

These equations can be solved by utilizing the fixed point iterations for L_i and H_i, $i = 1,\ 2,\ 3$ [15]. On the other hand, reduced-order pure-slow and pure-fast asymmetric algebraic Riccati equations are derived as follows.

$$P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0, \tag{26a}$$
$$P_{f1} b_1 - b_4 P_{f1} - b_3 + P_{f1} b_2 P_{f1} = 0, \tag{26b}$$
$$P_{f2} c_1 - c_4 P_{f2} - c_3 + P_{f2} c_2 P_{f2} = 0, \tag{26c}$$

where

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} := T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1,$$
$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} := T_{11}\varepsilon_1 L_1(T_{01} - T_{02}L_3), \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} := T_{22} + \varepsilon_2 L_2 T_{02}.$$

It should be noted that unique positive semidefinite stabilizing solutions exist for the asymmetric AREs defined in (26) exist. These solutions can be obtained by using Newton's method. It is well known that Newton's method converges quadratically under appropriate initial conditions. In fact, this important feature has been proved in [15]. Using the above results, the following matrix is defined.

$$\Pi := \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2^T K E_1, \tag{27}$$

where

$$K := \begin{bmatrix} I_{n_0} - \varepsilon_1 H_1 L_1 + \varepsilon_1 \varepsilon_2 H_1 H_2 L_2 + \varepsilon_2 H_3 L_2 \\ L_1 - \varepsilon_2 H_2 L_2 \\ L_2 \end{bmatrix.$$

$$\begin{matrix} -\varepsilon_1 H_1 + \varepsilon_1 \varepsilon_2 H_1 H_2 L_3 + \varepsilon_2 H_3 L_2 & \varepsilon_2 (H_3 + \varepsilon_1 H_1 H_2) \\ I_{n_1} - \varepsilon_2 H_2 L_3 & -\varepsilon_2 H_2 \\ L_3 & I_{n_2} \end{bmatrix},$$

$$E_1 := \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1^{-1} I_{n_1} & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_2^{-1} I_{n_2} \end{bmatrix},$$

$$E_2 := \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_2} \end{bmatrix}.$$

Finally, we can express P_ε in terms of P_s, P_{f1} and P_{f2}.

$$P_\varepsilon = \left[\Omega_3 + \Omega_4 \cdot \textbf{block diag}\begin{pmatrix} P_s & P_{f1} & P_{f2} \end{pmatrix}\right]$$

$$\cdot \left[\Omega_1 + \Omega_2 \cdot \textbf{block diag}\begin{pmatrix} P_s & P_{f1} & P_{f2} \end{pmatrix}\right]^{-1}, \tag{28}$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \Pi^{-1}.$$

However, these results are restricted to the MSPS such that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common (see e.g., Assumption 5, [17]). Thus, we cannot apply the technique proposed in [15] to the practical system.

3.4.2 Recursive Computation

Now, let us define $\phi := \|\varepsilon\| = \sqrt{\varepsilon_1^2 + \varepsilon_2^2}$. The solution (21) of MARE (8) can be changed as follows.

$$P_\varepsilon = \begin{bmatrix} \bar{P}_{00} + \phi E_{00} & \varepsilon_1(\bar{P}_{10} + \phi E_{10})^T & \varepsilon_2(\bar{P}_{20} + \phi E_{20})^T \\ \varepsilon_1(\bar{P}_{10} + \phi E_{10}) & \varepsilon_1(\bar{P}_{11} + \phi E_{11}) & \phi^2 E_{21}^T \\ \varepsilon_2(\bar{P}_{20} + \phi E_{20}) & \phi^2 E_{21} & \varepsilon_2(\bar{P}_{22} + \phi E_{22}) \end{bmatrix}, \qquad (29)$$

where $E_{00} = E_{00}^T$, $E_{11} = E_{11}^T$, $E_{22} = E_{22}^T$.

The $O(\|\varepsilon\|)$ approximation of the error terms E_{pq} will result in $O(\|\varepsilon\|^2)$ approximation of the required matrix P_{pq}. That is why we are interested in finding equations of the error terms and a convenient algorithm to find their solutions. Substituting (29) into (17), we arrive at the recursive algorithm.

$$\begin{aligned} & D_{11}^T E_{11}^{(n+1)} + E_{11}^{(n+1)} D_{11} \\ = \; & -\frac{\varepsilon_1}{\phi}(D_{01}^T \bar{P}_{10}^T + \bar{P}_{10} D_{01}) - \varepsilon_1(D_{01}^T E_{10}^{(n)T} + E_{10}^{(n)} D_{01}) + \frac{\varepsilon_1^2}{\phi} P_{10}^{(n)} S_{00} P_{10}^{(n)T} \\ & \quad + \varepsilon_1(E_{11}^{(n)} S_{01}^T P_{10}^{(n)T} + P_{10}^{(n)} S_{01} E_{11}^{(n)}) + \varepsilon_1\sqrt{\alpha_{12}}(E_{21}^{(n)T} S_{02}^T P_{10}^{(n)T} + \\ & + P_{10}^{(n)} S_{02} E_{21}^{(n)}) + \phi(E_{11}^{(n)} S_{11} E_{11}^{(n)} + \alpha_{12} E_{21}^{(n)T} S_{22} E_{21}^{(n)}), \end{aligned} \qquad (30a)$$

$$\begin{aligned} & D_{22}^T E_{22}^{(n+1)} + E_{22}^{(n+1)} D_{22} \\ = \; & -\frac{\varepsilon_2}{\phi}(D_{02}^T \bar{P}_{20}^T + \bar{P}_{20} D_{02}) - \varepsilon_2(D_{02}^T E_{20}^{(n)T} + E_{20}^{(n)} D_{02}) + \frac{\varepsilon_2^2}{\phi} P_{20}^{(n)} S_{00} P_{20}^{(n)T} \\ & \quad + \varepsilon_2(E_{22}^{(n)} S_{02}^T P_{20}^{(n)T} + P_{20}^{(n)} S_{02} E_{22}^{(n)}) + \frac{\varepsilon_2}{\sqrt{\alpha_{12}}}(E_{21}^{(n)} S_{01}^T P_{20}^{(n)T} + \\ & + P_{20}^{(n)} S_{01} E_{21}^{(n)T}) + \phi(E_{22}^{(n)} S_{22} E_{22}^{(n)} + \frac{1}{\alpha_{12}} E_{21}^{(n)} S_{11} E_{21}^{(n)T}), \end{aligned} \qquad (30b)$$

$$
\begin{aligned}
&\sqrt{\alpha_{12}}E_{21}^{(n+1)T}D_{22} + \frac{1}{\sqrt{\alpha_{12}}}D_{11}^T E_{21}^{(n+1)T} \\
= &-\frac{\varepsilon_1}{\phi}\bar{P}_{10}D_{02} - \frac{\varepsilon_2}{\phi}D_{01}^T\bar{P}_{20}^T - \varepsilon_1 E_{10}^{(n)}D_{02} - \varepsilon_2 D_{01}^T E_{20}^{(n)T} + \varepsilon_1(P_{10}^{(n)}S_{02}E_{22}^{(n)} + \\
&+\frac{1}{\sqrt{\alpha_{12}}}P_{10}^{(n)}S_{01}E_{21}^{(n)T}) + \varepsilon_2(E_{11}^{(n)}S_{01}^T P_{20}^{(n)T} + \sqrt{\alpha_{12}}E_{21}^{(n)T}S_{02}^T P_{20}^{(n)T}) \\
&+\frac{\varepsilon_1\varepsilon_2}{\phi}P_{10}^{(n)}S_{00}P_{20}^{(n)T} + \phi(\sqrt{\alpha_{12}}E_{21}^{(n)T}S_{22}E_{22}^{(n)} \qquad\qquad (30c)\\
&+\frac{1}{\sqrt{\alpha_{12}}}E_{11}^{(n)T}S_{11}^T E_{21}^{(n)T}),\\
&D_0^T E_{00}^{(n+1)} + E_{00}^{(n+1)}D_0 \\
= &-D_{10}^T D_{11}^{-T}H_{01}^{(n)T} - H_{01}^{(n)}D_{11}^{-1}D_{10} - D_{20}^T D_{22}^{-T}H_{02}^{(n)T} - H_{02}^{(n)}D_{22}^{-1}D_{20} \\
&+\phi(E_{00}^{(n)}S_{00}E_{00}^{(n)} + E_{10}^{(n)T}S_{01}^T E_{00}^{(n)} + E_{00}^{(n)}S_{01}E_{10}^{(n)} \\
&+E_{20}^{(n)T}S_{02}^T E_{00}^{(n)} + E_{00}^{(n)}S_{02}E_{20}^{(n)} + E_{10}^{(n)T}S_{11}^T E_{10}^{(n)} + E_{20}^{(n)T}S_{22}E_{20}^{(n)}), \qquad (30d)\\
&E_{i0}^{(n+1)T} = (H_{0i}^{(n)} - E_{00}^{(n+1)}D_{0i})D_{ii}^{-1},\ i = 1,\ 2, \qquad\qquad (30e)
\end{aligned}
$$

where

$$
\begin{aligned}
H_{01}^{(n)} &= -D_{10}^T E_{11}^{(n+1)} - \sqrt{\alpha_{12}}D_{20}^T E_{21}^{(n+1)} - \frac{\varepsilon_1}{\phi}D_{00}^T\bar{P}_{10}^T - \varepsilon_1 D_{00}^T E_{10}^{(n)T} + \\
&\quad +\phi(E_{00}^{(n)}S_{01}E_{11}^{(n)} + E_{10}^{(n)T}S_{11}E_{11}^{(n)}) + \phi\sqrt{\alpha_{12}}(E_{00}^{(n)}S_{02}E_{21}^{(n)} + \\
&+E_{20}^{(n)T}S_{22}^T E_{21}^{(n)}) + \varepsilon_1(E_{00}^{(n)}S_{00} + E_{10}^{(n)T}S_{01}^T + E_{20}^{(n)T}S_{02}^T)P_{10}^{(n)T},\\
H_{02}^{(n)} &= -D_{20}^T E_{22}^{(n+1)} - \frac{1}{\sqrt{\alpha_{12}}}D_{10}^T E_{21}^{(n+1)T} - \frac{\varepsilon_2}{\phi}D_{00}^T\bar{P}_{20}^T - \varepsilon_2 D_{00}^T E_{20}^{(n)T} + \\
&\quad +\phi(E_{00}^{(n)}S_{02}E_{22}^{(n)} + E_{20}^{(n)T}S_{22}E_{22}^{(n)}) + \frac{\phi}{\sqrt{\alpha_{12}}}(E_{00}^{(n)}S_{01}E_{21}^{(n)T} + \\
&+E_{10}^{(n)T}S_{11}^T E_{21}^{(n)T}) + \varepsilon_2(E_{00}^{(n)}S_{00} + +E_{10}^{(n)T}S_{01}^T + E_{20}^{(n)T}S_{02}^T)P_{20}^{(n)T},\\
P_{10}^{(n)} &= \bar{P}_{10} + \phi E_{10}^{(n)},\quad P_{20}^{(n)} = \bar{P}_{20} + \phi E_{20}^{(n)},\\
E_{00}^{(0)} &= E_{10}^{(0)} = E_{20}^{(0)} = E_{11}^{(0)} = E_{21}^{(0)} = E_{22}^{(0)} = 0.
\end{aligned}
$$

The following theorem indicates the convergence of the algorithm (30).

Theorem 5. *[49] Under Assumptions 1 and 2, there exist the unique and bounded solutions E_{pq} of the error equation in a neighborhood of $\|\varepsilon\| = 0$.*

Moreover, the algorithm (30) *converges to the exact solution* E_{pq} *with the rate of convergence of* $O(\|\varepsilon\|^n)$*, that is*

$$\|E_{pq} - E_{pq}^{(n)}\| = O(\|\varepsilon\|^n), \ n = 1, \ 2, \ \dots , \ pq = 00, \ 10, \ 20, \ 11, \ 21, \ 22. \quad (31)$$

However, there exists the drawback that the recursive algorithm converges only to the approximation solution [49] since the convergence of the recursive algorithm depends on the zeroth-order solutions.

3.4.3 Newton's Method

In this section, we develop an elegant and simple algorithm which converges globally to the positive semidefinite solution of the MARE (8). The algorithm uses the Kleinman algorithm [33], which is equivalent to Newton's method. Thus, this paper presents important improvements upon some of the results of [15, 49] in the sense that one need not assume that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common. Moreover, the convergence solution does not depend on the initial guess, and quadratic convergence is attained.

We propose the following algorithm for solving the MARE (8)

$$(A - SP^{(n)})^T P^{(n+1)} + P^{(n+1)T}(A - SP^{(n)}) + P^{(n)T} SP^{(n)} + Q = 0, \quad (32)$$

$$i = 0, \ 1, \ 2, \ \dots , \ P_\varepsilon^{(n)} = \Phi_\varepsilon P^{(n)} = P^{(n)T}\Phi_\varepsilon,$$

$$P^{(n)} = \begin{bmatrix} P_{00}^{(n)} & \varepsilon_1 P_{10}^{(n)T} & \varepsilon_2 \ P_{20}^{(n)T} \\ P_{10}^{(n)} & P_{11}^{(n)} & \dfrac{1}{\sqrt{\alpha_{21}}} P_{21}^{(n)T} \\ P_{20}^{(n)} & \sqrt{\alpha_{21}} P_{21}^{(n)} & P_{22}^{(n)} \end{bmatrix}, \ A = \Phi_\varepsilon A_\varepsilon, \ S = \Phi_\varepsilon S_\varepsilon \Phi_\varepsilon$$

with the initial condition

$$P^{(0)} = \begin{bmatrix} \bar{P}_{00} & \varepsilon_1 \bar{P}_{10} T & \varepsilon_2 \bar{P}_{20}^T \\ \bar{P}_{10} & \bar{P}_{11} & 0 \\ \bar{P}_{20} & 0 & \bar{P}_{22} \end{bmatrix}, \quad (33)$$

where $\bar{P}_{pq}, \ pq = 00, \ 10, \ 20, \ 11, \ 22$ are defined by (19).

The algorithm (32) has the feature given in the following theorem.

Theorem 6. *[50] Under Assumptions 1 and 2, there exists a positive scalar* $\tilde{\sigma}_1$ *such that for all* $\varepsilon \in H$ *with* $0 < \|\varepsilon\| \le \tilde{\sigma}_1$ *the iterative algorithm* (32) *converges to the exact solution* $P_\varepsilon^* = \Phi_\varepsilon P^* = P^{*T}\Phi_\varepsilon$ *with the rate of quadratic* *convergence, where* $P_\varepsilon^{(n)} = \Phi_\varepsilon P^{(n)} = P^{(n)T}\Phi_\varepsilon$ *is positive semidefinite. Moreover, zero-order solution* $P^{(0)}$ *is in the neighborhood of the exact solution* P_ε^**. That is, the following conditions are satisfied.*

$$\|P^{(n)} - P^*\| \le \frac{(2\theta)^{2^n}}{2^n \beta \boldsymbol{L}} = O(\|\varepsilon\|^{2^n}),\ n = 0,\ 1,\ 2,\ \cdots, \tag{34a}$$

$$\|P^{(0)} - P^*\| \le \frac{1}{\beta \boldsymbol{L}}[1 - \sqrt{1-2\theta}], \tag{34b}$$

where

$$\boldsymbol{L} := 2\|S\| < \infty,\ \beta := \|[\nabla \boldsymbol{F}(\boldsymbol{P}_0)]^{-1}\|,\ \theta := \beta\eta\boldsymbol{L}$$

with

$$\eta := \beta \cdot \|\boldsymbol{F}(\boldsymbol{P}_0)\|,\ \boldsymbol{F}(\boldsymbol{P}) := \begin{bmatrix} \mathrm{vec}F_{00} \\ \mathrm{vec}F_{10} \\ \mathrm{vec}F_{20} \\ \mathrm{vec}F_{11} \\ \mathrm{vec}F_{21} \\ \mathrm{vec}F_{22} \end{bmatrix},$$

$$A^TP + P^TA - P^TSP + Q = \begin{bmatrix} F_{00} & F_{10}^T & F_{20}^T \\ F_{10} & F_{11} & F_{21}^T \\ F_{20} & F_{21} & F_{22} \end{bmatrix},$$

and

$$\nabla\boldsymbol{F}(\boldsymbol{P}) := \frac{\partial \boldsymbol{F}(\boldsymbol{P})}{\partial \boldsymbol{P}^T},\ \boldsymbol{P} = \begin{bmatrix} \mathrm{vec}P_{00} \\ \mathrm{vec}P_{10} \\ \mathrm{vec}P_{20} \\ \mathrm{vec}P_{11} \\ \mathrm{vec}P_{21} \\ \mathrm{vec}P_{22} \end{bmatrix},\ \boldsymbol{P}_0 = \begin{bmatrix} \mathrm{vec}\bar{P}_{00} \\ \mathrm{vec}\bar{P}_{10} \\ \mathrm{vec}\bar{P}_{20} \\ \mathrm{vec}\bar{P}_{11} \\ 0 \\ \mathrm{vec}\bar{P}_{22} \end{bmatrix}.$$

These proofs can be derived by applying the Newton-Kantorovich theorem [34, 35].

It should be noted that the proposed algorithm, which is based on the Kleinman algorithm, has quadratic convergence. It may also be noted that to solve the multiparameter algebraic Lyapunov equation (MALE), a fixed-point algorithm can be combined. See [50] for details. In addition, it has been proved that the resulting $O(\|\varepsilon\|^{2^n})$ accuracy controller achieves the cost $J_{\rm opt} + O(\|\varepsilon\|^{2^{n+1}})$.

Remark 2. *Using the Newton-Kantorovich theorem [34, 35], which will be presented later in this paper, it is clear that there exists a positive scalar* $\tilde{\sigma}_2$ *such that for all* $\varepsilon \in H$ *with* $0 < \|\varepsilon\| \leq \tilde{\sigma}_2$*, the MARE* (8) *has positive semidefinite solutions within the limits of the sufficiency condition. Moreover, it should be noted that the asymptotic structure of* (21) *can also be obtained by applying the Newton-Kantorovich theorem.*

4 Extension to Other Problem

The above-mentioned techniques can be demonstrated for the filtering and the various control.

4.1 Filtering Problem

Filtering problems for MSPS have been investigated extensively. In [51], a new design method for the near-optimal Kalman filters has been proposed. As a result, the high-dimensional ill-conditioned MARE is replaced by the low-order singular perturbation parameter-independent ARE. Furthermore, the proposed filters can be implemented even if the fast state matrices are singular and the perturbation parameters are unknown. In [12], the well-posedness of multimodel strategies for a LQ-Gaussian (LQG) optimal control problem has been studied. In addition, numerical stiffness is avoided by using the exact slow-fast decomposition method for solving the filtered MARE in [17]. The local control problem of a control agent of the above paper is obtained by neglecting the fast dynamics of the other agent's subsystem, and each agent uses the optimal solution of its local control problem. However, the nonsingularity assumptions for the fast state matrices A_{ii}, $i = 1, \ldots, N$ are also needed. To avoid this drawback, a new recursive algorithm for solving the MARE has been proposed [54]. It has been proved that the solution of the MARE converges to a positive semi-definite stabilizing solution with the rate of convergence of $O(\|\varepsilon\|^{n+1})$, where i denotes the number of required iterations. Moreover, it has been recently proved that the resulting Kalman filter achieves a performance level, i.e. $O(\|\varepsilon\|^{2n+1})$, close to the optimal mean square error.

4.2 H_∞ Control Problem

The asymptotic expansions for MARE with a sign-indefinite quadratic term that arises in the H_∞ control problem and an iterative technique for solving such MARE are described in [48]. In [59], a new iterative algorithm for solving MARE with a sign-indefinite quadratic term has been proposed for the general case. The proposed algorithm consists of Newton's method and two fixed-point algorithms. As a result, it has been proven that the solution of the MARE converges to a positive semi-definite stabilizing solution with a rate of convergence of $O(\|\varepsilon\|^{2^n})$. Moreover, compared with the existing results [48], a reduction in the size of the computational work space can be achieved even if the MSPS has many fast subsystems. This algorithm for solving the MARE and MALE is applied to a wide class of control law synthesis methods involving a solution to the MARE, such as in the robust stabilizing control problem. On the other hand, a reliable H_∞ control for linear time-invariant MSPS against sensor failures has been investigated [30]. The main contribution of this paper was an extension of the previous study of the reliable H_∞ control.

4.3 Guaranteed Cost Control Problem

The multi-parameter singularly perturbed guaranteed cost control problem has been demonstrated [56]. By solving the reduced-order slow and fast AREs, the new ε-independent guaranteed cost controller can be obtained. The new technique has the following advantages: It does not need information on the small parameters ε_i. The required work space is the same as that of the reduced-order slow and fast subsystems. The present new results can be applied to the MSPS without the need for the various assumptions that have been made for the fast subsystems in the existing results, although the fast subsystems have the uncertainty. Therefore, the new design approach has been successfully applied to a more practical uncertain MSPS. Furthermore, if the parameters are known, we can obtain the exact GCC by using the above-mentioned numerical technique. As another important approach to the uncertain MSPS except for the guaranteed cost control problem, the fault diagnosis of two-time-scale MSPSs has been considered in [31].

5 Nash Games

The LQ Nash games for the MSPS have been studied by using composite controller design [5, 57, 58]. Furthermore, a decentralized stochastic Nash game has been presented for two decision makers controlling MSPS [8]. According to this result, in order to obtain near-equilibrium Nash strategies, the decision makers need only to solve two coupled low-order stochastic control problems. Furthermore, decentralized team strategies for decision makers using MSPS have been developed [10]. The well-posedness of the multimodel solution was demonstrated. Recently, computational approaches for Nash games have been studied [53, 55, 62]. For obtaining the strategies, Newton's method [55] seems to be very powerful tool. In this section, existing and recent progress on the use of the two-time-scale decomposition method and numerical analysis related to Nash games for MSPSs will be reviewed.

5.1 Parameter Independent Strategies

Consider a linear time-invariant MSPS

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{N} A_{0j} z_j(t) + \sum_{j=1}^{N} B_{0j} u_j(t),\ x(0) = x^0, \tag{35a}$$

$$\varepsilon_i \dot{z}_i(t) = A_{i0} x(t) + A_{ii} z_i(t) + B_{ii} u_i(t),\ z_i(0) = z_i^0,\ i = 1,\ \dots, N, \tag{35b}$$

with the quadratic cost functions

$$J_i(u_1,\ \dots, u_N) = \frac{1}{2}\int_0^{\infty} [y_i^T y_i + u_i^T R_{ii} u_i] dt, \tag{36a}$$

$$y_i = C_{i0} x + C_{ii} z_i = C_i \xi. \tag{36b}$$

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. Our purpose is to find a linear feedback strategy set $(u_1^*,\ \dots, u_N^*)$ such that

$$J_i(u_1^*,\ \dots,\ u_N^*) \le J_i(u_1^*,\ \dots,\ u_{i-1}^*,\ u_i,\ u_{i+1}^*,\ \dots,\ u_N^*),\ i = 1,\ \dots, N \tag{37}$$

The decision makers are required to select the closed loop strategy u_i^*, if they exist, such that (37) holds. Moreover, each player uses the strategy u_i^* such that the closed-loop system is asymptotically stable for sufficiently small ε_i. The following lemma is already known [36].

Lemma 5. *There exists an admissible strategy such that the inequality* (37) *holds iff the cross-coupled multiparameter algebraic Riccati equations* (CMAREs)

$$P_{i\varepsilon}\left(A_\varepsilon-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}\right)+\left(A_\varepsilon-\sum_{j=1}^{N}S_{j\varepsilon}P_{j\varepsilon}\right)^{T}P_{i\varepsilon}+P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon}+Q_i=0, \quad (38)$$

$i=1,\ \ldots\ ,N$,*have solutions* $P_{i\varepsilon}\geq 0$, *where*

$$P_{i\varepsilon}:=\begin{bmatrix} P_{i00} & P_{if0}^{T}\Pi_\varepsilon \\ \Pi_\varepsilon P_{if0} & \Pi_\varepsilon P_{if}\end{bmatrix},\ P_{i00}=P_{i00}^{T},\ P_{if0}\begin{bmatrix} P_{i10} \\ \vdots \\ P_{iN0}\end{bmatrix},$$

$$P_{if}:=\begin{bmatrix} P_{i11} & \alpha_{12}P_{i21}^{T} & \alpha_{13}P_{i31}^{T} & \cdots & \alpha_{1N}P_{iN1}^{T} \\ P_{i21} & P_{i22} & \alpha_{23}P_{i32}^{T} & \cdots & \alpha_{2N}P_{iN2}^{T} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{i(N-1)1} & P_{i(N-1)2} & P_{i(N-1)3} & \cdots & \alpha_{(N-1)N}P_{iN(N-1)}^{T} \\ P_{iN1} & P_{iN2} & P_{iN3} & \cdots & P_{iNN}\end{bmatrix},$$

$$B_1:=\begin{bmatrix} B_{10} \\ B_{11} \\ 0 \\ \vdots \\ 0\end{bmatrix},\ \cdots\ ,B_i:=\begin{bmatrix} B_{i0} \\ \vdots \\ B_{ii} \\ \vdots \\ 0\end{bmatrix},\ \cdots\ ,B_N:=\begin{bmatrix} B_{0N} \\ 0 \\ 0 \\ \vdots \\ B_{NN}\end{bmatrix},$$

$$S_{i\varepsilon}:=\Phi_\varepsilon^{-1}B_iR_{ii}^{-1}B_i^{T}\Phi_\varepsilon^{-1},$$

$$S_i:=B_iR_{ii}^{-1}B_i^{T}=\begin{bmatrix} S_{i00} & O & S_{i0i} & O \\ O & O & O & O \\ S_{i0i}^{T} & O & S_{iii} & O \\ O & O & O & O\end{bmatrix},$$

$$Q_i:=C_iC_i^{T}=\begin{bmatrix} Q_{i00} & O & Q_{i0i} & O \\ O & O & O & O \\ Q_{i0i}^{T} & O & Q_{iii} & O \\ O & O & O & O\end{bmatrix},$$

$$\Phi_\varepsilon:=\boldsymbol{b}lock\ diag\begin{pmatrix} I_{n_0} & \varepsilon_1 I_{n_1} & \cdots & \varepsilon_N I_{n_N}\end{pmatrix}.$$

Then the closed-loop linear Nash equilibrium solutions to the full-order prob-

lem are given by

$$u_i^*(t) = -R_{ii}^{-1} B_{i\varepsilon}^T P_{i\varepsilon} \xi(t). \tag{39}$$

It should be noted that it is impossible to solve the CMARE (38) if the small perturbed parameter ε_i are unknown. Thus, the purpose of this section is to find the parameter-independent Nash strategies.

The parameter-independent Nash strategies for the MSPS will be studied under the following basic assumption.

Assumption 3. *The Hamiltonian matrices T_{iii}, $i = 1,\ \dots\ N$ are nonsingular, where*

$$T_{iii} := \begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{iii} & -A_{ii}^T \end{bmatrix}. \tag{40}$$

Under Assumptions 1-3, the following zeroth-order equations of the CMAREs (38) are given as $\|\varepsilon\| \to +0$.

$$\begin{aligned}
&\bar{P}_{i00}\left(A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}\right) + \left(A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}\right)^T \bar{P}_{i00} + && \text{(41a)}\\
&+\bar{P}_{i00}S_{s_i}\bar{P}_{i00} + Q_{s_i} = 0, &&\\
&A_{ii}^T\bar{P}_{iii} + \bar{P}_{iii}A_{ii} - \bar{P}_{iii}S_{iii}\bar{P}_{iii} + Q_{iii} = 0, && \text{(41b)}\\
&\bar{P}_{ikl} = 0,\ k > l,\ \bar{P}_{ijj} = 0,\ i \neq j && \text{(41c)}\\
&\quad \begin{bmatrix} \bar{P}_{110} & \bar{P}_{210} & \cdots & \bar{P}_{N10} \end{bmatrix} = \begin{bmatrix} \bar{P}_{111} \\ -I_{n_1} \end{bmatrix}^T T_{111}^{-1} T_{110} \begin{bmatrix} I_{n_0} & 0 & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}, &&\\
&\quad \begin{bmatrix} \bar{P}_{120} & \bar{P}_{220} & \cdots & \bar{P}_{N20} \end{bmatrix} = \begin{bmatrix} \bar{P}_{222} \\ -I_{n_2} \end{bmatrix}^T T_{222}^{-1} T_{220} \begin{bmatrix} 0 & I_{n_0} & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}, &&\\
&\quad \vdots &&\\
&\quad \begin{bmatrix} \bar{P}_{1N0} & \bar{P}_{2N0} \cdots \bar{P}_{NN0} \end{bmatrix} = \begin{bmatrix} \bar{P}_{NNN} \\ -I_{n_N} \end{bmatrix}^T T_{NNN}^{-1} T_{NN0} \begin{bmatrix} 0 & 0 & \cdots & I_{n_0} \\ \bar{P}_{100} & \bar{P}_{200} \cdots & \bar{P}_{N00} \end{bmatrix}, && \text{(41d)}
\end{aligned}$$

where

$$\begin{bmatrix} A_s & * \\ * & -A_s^T \end{bmatrix} = \begin{bmatrix} A_0 & * \\ * & -A_0^T \end{bmatrix} - \sum_{i=1}^{N} T_{i0i} T_{iii}^{-1} T_{ii0},$$

$$\begin{bmatrix} * & -S_{s_i} \\ -Q_{s_i} & * \end{bmatrix} = T_{i00} - T_{i0i} T_{iii}^{-1} T_{ii0},$$

$$T_{i00} = \begin{bmatrix} A_0 & -S_{i00} \\ -Q_{i00} & -A_0^T \end{bmatrix}, T_{i0i} = \begin{bmatrix} A_{0i} & -S_{i0i} \\ -Q_{i0i} & -A_{i0}^T \end{bmatrix}, \ T_{ii0} = \begin{bmatrix} A_{i0} & -S_{i0i}^T \\ -Q_{i0i}^T & -A_{0i}^T \end{bmatrix},$$
$$i = 1, \ \dots, N.$$

The following theorem shows the relation between the solutions P_i and the zeroth-order solutions $\bar{P}_{ikl}$ $i = 1, \ \dots, N, \ k \geq l, \ 0 \leq k, \ l \leq N$.

$$\det \begin{bmatrix} \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & -(S_{s_2}\bar{P}_{100}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_2}\bar{P}_{100}) & \cdots & -(S_{s_N}\bar{P}_{100}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_N}\bar{P}_{100}) \\ -(S_{s_1}\bar{P}_{200}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_1}\bar{P}_{200}) & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & \cdots & -(S_{s_N}\bar{P}_{200}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_N}\bar{P}_{200}) \\ \vdots & \vdots & \ddots & \vdots \\ -(S_{s_1}\bar{P}_{N00}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_1}\bar{P}_{N00}) & -(S_{s_2}\bar{P}_{N00}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_2}\bar{P}_{N00}) & \cdots & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T \end{bmatrix} \neq 0, \quad (42)$$

where $\hat{A}_s := A_s - \sum_{j=1}^{N} S_{s_j} \bar{P}_{j00}$ and $\hat{A}_s$ are stable matrix.

Theorem 7. *Suppose that the condition* (42) *holds. Under Assumptions 1 and 2, there is a neighborhood* $\boldsymbol{V}(0)$ *of* $\|\varepsilon\| = 0$ *such that for all* $\|\varepsilon\| \in \boldsymbol{V}(0)$ *there exists a solution* $P_i = P_i(\varepsilon_1, \ \dots, \varepsilon_N)$. *These solutions are unique in a neighborhood of* $\bar{P}_i = P_i(0, \ \dots, 0)$. *Then, the MARE* (38) *possess the power series expansion at* $\|\varepsilon\| = 0$. *That is, the following form is satisfied.*

$$P_{i\varepsilon} := \Phi_\varepsilon P_i, P_i = \bar{P}_i + O(\|\varepsilon\|) = \begin{bmatrix} \bar{P}_{i00} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \bar{P}_{i10} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{ii0} & 0 & \cdots & 0 & \bar{P}_{iii} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{iN0} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + O(\|\varepsilon\|). \quad (43)$$

5.2 Numerical Algorithms

When the parameters represent small unknown perturbations whose values are not known exactly, the previously introduced composite design is very useful. However, the composite Nash equilibrium solution achieves only a performance level of $O(\|\varepsilon\|)$, close to the full-order performance. Another important drawback is that since the closed-loop solution of the reduced Nash problem depends on the path along $\varepsilon_1/\varepsilon_2$ as $\|\varepsilon\| \to +0$, we cannot conclude that the closed-loop solution of the full problem converges to the closed-loop solution of the reduced problem [2]. Therefore, as long as the small perturbation parameters ε_i are known, much effort should be made towards finding the exact strategies which guarantees Nash equilibrium without ill-conditioning. In this subsection, the iterative algorithms for solving the CMAREs are summarized.

5.2.1 Recursive Computation

A recursive algorithm for solving singularly perturbed Nash games has been attempted [53]. It has been shown that the recursive algorithm is very effective in solving the CMAREs when the system matrices are functions of a small perturbation parameter ε_i. However, the recursive algorithm converges only to the approximation solution because the convergence solutions depend on the zeroth-order solutions. In addition, the recursive algorithm has the property of linear convergence. Thus, the convergence speed is very slow.

5.2.2 Newton's Method

In order to improve the convergence rate of the recursive algorithm, we propose the following algorithm which is based on the Newton's method.

$$\Phi^{(n)T}\boldsymbol{P}^{(n+1)} + \boldsymbol{P}^{(n+1)T}\Phi^{(n)} - \Theta^{(n)T}\boldsymbol{P}^{(n+1)}\boldsymbol{J} - \boldsymbol{J}\boldsymbol{P}^{(n+1)T}\Theta^{(n)} + \Xi^{(n)} = 0,$$
$$n = 0,\ 1,\ \ldots\ , \tag{44}$$
$$\Leftrightarrow \begin{cases} \Phi_1^{(n)T}P_1^{(n+1)} + P_1^{(n+1)T}\Phi_1^{(n)} - \Theta_2^{(n)T}P_2^{(n+1)} - P_2^{(n+1)T}\Theta_2^{(n)} + \Xi_1^{(n)} = 0, \\ \Phi_2^{(n)T}P_2^{(n+1)} + P_2^{(n+1)T}\Phi_2^{(n)} - \Theta_1^{(n)T}P_1^{(n+1)} - P_1^{(n+1)T}\Theta_1^{(n)} + \Xi_2^{(n)} = 0, \end{cases}$$

where

$$\Phi^{(n)} := \tilde{A} - \tilde{S}\boldsymbol{P}^{(n)} - \boldsymbol{J}\tilde{S}\boldsymbol{P}^{(n)}\boldsymbol{J} = \begin{bmatrix} \Phi_1^{(n)} & 0 \\ 0 & \Phi_2^{(n)} \end{bmatrix},$$

$$\begin{aligned}
\Theta^{(n)} &:= \tilde{S}\boldsymbol{J}\boldsymbol{P}^{(n)} = \begin{bmatrix} 0 & \Theta_1^{(n)} \\ \Theta_2^{(n)} & 0 \end{bmatrix}, \\
\Xi^{(n)} &:= \tilde{Q} + \boldsymbol{P}^{(n)T}\tilde{S}\boldsymbol{P}^{(n)} + \boldsymbol{J}\boldsymbol{P}^{(n)T}\tilde{S}\boldsymbol{J}\boldsymbol{P}^{(n)} + \boldsymbol{P}^{(n)T}\boldsymbol{J}\tilde{S}\boldsymbol{P}^{(n)}\boldsymbol{J} \\
&= \begin{bmatrix} \Xi_1^{(n)} & 0 \\ 0 & \Xi_2^{(n)} \end{bmatrix}, \\
\Phi_i^{(n)} &:= \begin{bmatrix} \Phi_{00i}^{(n)} & \Phi_{01i}^{(n)} & \Phi_{02i}^{(n)} \\ \Phi_{10i}^{(n)} & \Phi_{11i}^{(n)} & \Phi_{12i}^{(n)} \\ \Phi_{20i}^{(n)} & \Phi_{21i}^{(n)} & \Phi_{22i}^{(n)} \end{bmatrix}, \ \Theta_i^{(n)} := \begin{bmatrix} \Theta_{00i}^{(n)} & \Theta_{01i}^{(n)} & \Theta_{02i}^{(n)} \\ \Theta_{10i}^{(n)} & \Theta_{11i}^{(n)} & \Theta_{12i}^{(n)} \\ \Theta_{20i}^{(n)} & \Theta_{21i}^{(n)} & \Theta_{22i}^{(n)} \end{bmatrix}, \\
\Xi_i^{(n)} &:= \begin{bmatrix} \Xi_{00i}^{(n)} & \Xi_{01i}^{(n)} & \Xi_{02i}^{(n)} \\ \Xi_{01i}^{(n)T} & \Xi_{11i}^{(n)} & \Xi_{12i}^{(n)} \\ \Xi_{02i}^{(n)T} & \Xi_{12i}^{(n)T} & \Xi_{22i}^{(n)} \end{bmatrix}, \ i = 1,\ 2, \\
\boldsymbol{P}^{(n)} &:= \begin{bmatrix} P_1^{(n)} & 0 \\ 0 & P_2^{(n)} \end{bmatrix}, \\
P_1^{(n)} &:= \begin{bmatrix} P_{100}^{(n)} & \varepsilon_1 P_{110}^{(n)T} & \varepsilon_2 P_{120}^{(n)T} \\ P_{110}^{(n)} & P_{111}^{(n)} & \sqrt{\alpha_{21}}^{-1} P_{121}^{(n)T} \\ P_{120}^{(n)} & \sqrt{\alpha_{21}} P_{121}^{(n)} & P_{122}^{(n)} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
P_2^{(n)} &:= \begin{bmatrix} P_{200}^{(n)} & \varepsilon_1 P_{210}^{(n)T} & \varepsilon_2 P_{220}^{(n)T} \\ P_{210}^{(n)} & P_{211}^{(n)} & \sqrt{\alpha_{21}}^{-1} P_{221}^{(n)T} \\ P_{220}^{(n)} & \sqrt{\alpha_{21}} P_{221}^{(n)} & P_{222}^{(n)} \end{bmatrix}, \\
\tilde{A} &:= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \ \tilde{Q} := \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \ \tilde{S} := \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \\
\boldsymbol{J} &:= \begin{bmatrix} 0 & I_{\bar{n}} \\ I_{\bar{n}} & 0 \end{bmatrix}, \ A := \Phi_\varepsilon A_\varepsilon.
\end{aligned}$$

and the initial condition $\boldsymbol{P}^{(0)}$ has the following form

$$\boldsymbol{P}^{(0)} = \begin{bmatrix} P_1^{(0)} & 0 \\ 0 & P_2^{(0)} \end{bmatrix} = \begin{bmatrix} \bar{P}_{100} & \varepsilon_1 \bar{P}_{110}^T & \varepsilon_2 \bar{P}_{120}^T & 0 & 0 & 0 \\ \bar{P}_{110} & \bar{P}_{111} & 0 & 0 & 0 & 0 \\ \bar{P}_{120} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{P}_{200} & \varepsilon_1 \bar{P}_{210}^T & \varepsilon_2 \bar{P}_{220}^T \\ 0 & 0 & 0 & \bar{P}_{210} & 0 & 0 \\ 0 & 0 & 0 & \bar{P}_{220} & 0 & \bar{P}_{222} \end{bmatrix}. \tag{45}$$

Note that the considered algorithm (44) is original. The new algorithm (44) can be constructed by setting $\boldsymbol{P}^{(n+1)} = \boldsymbol{P}^{(n)} + \Delta\boldsymbol{P}^{(n)}$ and neglecting $O(\Delta\boldsymbol{P}^{(n)T}\Delta\boldsymbol{P}^{(n)})$ term. Newton's method is well-known and is widely used to find a solution of the algebraic equations, and its local convergence properties are well understood.

Theorem 8. *Under Assumptions 1-3, the new iterative algorithm* (44) *converges to the exact solution* $\boldsymbol{P}^*$ *of the CMAREs* (38) *with the rate of quadratic convergence. Furthermore, the unique bounded solution* $\boldsymbol{P}^{(n)}$ *of the CMAREs* (38) *is in the neighborhood of the exact solution* $\boldsymbol{P}^*$. *That is, the following conditions are satisfied.*

$$\|\boldsymbol{P}^{(n)} - \boldsymbol{P}^*\| \le O(\|\varepsilon\|^{2^n}),\ n = 0,\ 1,\ \ldots\ , \tag{46a}$$

$$\|\boldsymbol{P}^{(n)} - \boldsymbol{P}^*\| \le \frac{1}{\tilde{\beta}\tilde{\boldsymbol{L}}}[1 - \sqrt{1 - 2\tilde{\theta}}],\ n = 0,\ 1,\ \ldots\ , \tag{46b}$$

where

$$\boldsymbol{P} = \boldsymbol{P}^* = \begin{bmatrix} P_1^* & 0 \\ 0 & P_2^* \end{bmatrix},\ \tilde{\boldsymbol{L}} := 6\|\tilde{S}\|,\ \tilde{\beta} := \|[\nabla\boldsymbol{F}(\boldsymbol{P}^{(0)})]^{-1}\|,\ \tilde{\theta} := \tilde{\beta}\tilde{\eta}\tilde{\boldsymbol{L}},$$
$$\tilde{\eta} := \|[\nabla\boldsymbol{F}(\boldsymbol{P}^{(0)})]^{-1}\| \cdot \|\boldsymbol{F}(\boldsymbol{P}^{(0)})\|.$$

6 Stochastic MSPS Governed by Itô Equations

The various control problems for stochastic systems governed by Itô's differential equation have attracted considerable research interest. The stabilization, LQ optimal control and H_∞ control problems for singularly perturbed stochastic systems (SPSS) with state-dependent noise were investigated [37, 43, 44]. Although these results are very elegant and despite it being easy to obtain a controller, the multiparameter singularly perturbed stochastic systems (MSPSS) remain to be considered. The problem of exponential stability of the zero state equilibrium of a linear stochastic system modeled by a system of singularly perturbed Itô differential equations is investigated in [20, 37, 42],

The LQ optimal stochastic control problem for MSPSS in which N lower-level fast subsystems are interconnected through a higher-level slow subsystem has been investigated [60]. The stochastic H_∞ control problem for the MSPSS has been discussed [61]. In particular, a new iterative algorithm for solving the stochastic multimodeling algebraic Riccati equation (SMARE) that has sign-indefinite quadratic form has been proposed. Stochastic Nash games have been studied for stochastic multimodeling systems [62]. The main contribution of this paper is the new strategy set that is independent of the small parameters. In [63], the guaranteed cost control problem for MSPSS has been re-formulated as an extension of [56].

In this section, the numerical solution to the SMARE with a sign-indefinite quadratic term related to the stochastic H_∞ control problem with state-dependent noise is investigated. It may be noted that a similar technique can be used for several stochastic control problems [60, 62, 63].

We consider the following MSPSS that consist of N-fast subsystems with specific structure of lower level interconnected through the dynamics of a higher level slow subsystem.

$$d\xi(t) = [A_\varepsilon \xi(t) + B_\varepsilon u(t) + D_\varepsilon v(t)]dt + \sum_{p=1}^{M} A_{p\varepsilon}\xi(t)dw_p(t), \tag{47a}$$

$$z(t) = \begin{bmatrix} C\xi(t) \\ Hu(t) \end{bmatrix}, \tag{47b}$$

where

$$\begin{aligned}
\xi(t) &:= \begin{bmatrix} x(t) \\ z_1(t) \\ \vdots \\ z_N(t) \end{bmatrix} \in \Re^{\bar{n}},\ u(t) := \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix} \in \Re^{\bar{m}}, \\
v(t) &:= \begin{bmatrix} v_1(t) \\ \vdots \\ v_N(t) \end{bmatrix} \in \Re^{\bar{l}}, \\
\bar{n} &:= \sum_{j=0}^{N} n_j,\ \bar{m} := \sum_{j=1}^{N} m_j,\ \bar{l} := \sum_{j=1}^{N} l_j, \\
A_{p\varepsilon} &:= \begin{bmatrix} A_{p0} & \mu A_{p0f} \\ \Pi_\varepsilon^{-1}\bar{\varepsilon}^\delta A_{pf0} & \Pi_\varepsilon^{-1}\bar{\varepsilon}^\delta A_{pf} \end{bmatrix},\ A_{p0f} := \begin{bmatrix} A_{p01} & \cdots & A_{p0N} \end{bmatrix}, \\
A_{pf0} &:= \begin{bmatrix} A_{p10}^T & \cdots & A_{pN0}^T \end{bmatrix}^T, \\
A_{pf} &:= \textbf{block diag}\begin{pmatrix} A_{p11} & \cdots & A_{pNN} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned} D_\varepsilon &:= \begin{bmatrix} D_0 \\ \Pi_\varepsilon^{-1} D_f \end{bmatrix}, \ D_0 := \begin{bmatrix} D_{01} & \cdots & D_{0N} \end{bmatrix}, \\ D_f &:= \textbf{block diag} \begin{pmatrix} D_{11} & \cdots & D_{NN} \end{pmatrix}, \\ H &:= \textbf{block diag} \begin{pmatrix} H_{11} & \cdots & H_{NN} \end{pmatrix}. \end{aligned}$$

$v_i(t) \in L_2^{l_i}(0, \infty)$, $i = 1, \dots, N$ is considered to be an unknown finite-energy deterministic disturbance [45, 46]. $z(t) \in \Re^p$ is the controlled output. $\varepsilon_i > 0$, $i = 1, \dots, N$ and $\mu > 0$ are small parameters and $\delta > 1/2$ is independent of $\bar{\varepsilon} := \min\{\varepsilon_1, \dots, \varepsilon_N\}$. It should be noted that the parameters μ and δ have been introduced in [43, 44] for the first time. Moreover, the considered MSPSS consists of N-fast subsystems as compared to [43]. $w_p(t) \in \Re$, $p = 1, \dots, M$ is a one-dimensional standard Wiener process defined in the filtered probability space. Note that one of the fast state matrices A_{ii}, $i = 1, \dots, N$ may be singular.

Remark 3. *In stochastic problems, careful treatment is required to establish the validity of the multimodel problem [11]. In addition to the usual difficulties encountered in modeling a fast stochastic variable, the problem is rether involved due to the presence of information patterns. To simplify this aspect, the scaling parameter μ is considered.*

Without loss of generality, the stochastic H_∞ control problem for the MSPSS is investigated under the following basic assumption [45, 46].

Assumption 4. $H^T H = I_{\bar{m}}$.

It should be noted that the matrix pair (E, G) is deemed stable, if $d\xi(t) = E\xi(t)dt + G\xi(t)dw$ is asymptotically mean square stable [46].

The stochastic H_∞ control problem for MSPSS is given below [45, 46].

Given a constant $\gamma > 0$, find a matrix K satisfying the following conditions:

i) *The system*

$$d\xi(t) = [A_\varepsilon + B_\varepsilon K]\xi(t)dt + \sum_{p=1}^{M} A_{p\varepsilon}\xi(t)dw_p(t) \tag{48}$$

is exponentially mean-square stable (EMSS) *internally, i.e. it satisfies the following equation.*

$$E\|\xi(t)\|^2 \leq \rho e^{-\psi(t-s)} E\|\xi(s)\|^2, \ \exists \rho, \ \psi > 0. \tag{49}$$

ii) *The closed-loop system*

$$d\xi(t) = [(A_\varepsilon + B_\varepsilon K)\xi(t) + D_\varepsilon v(t)]dt + \sum_{p=1}^{M} A_{p\varepsilon}\xi(t)dw_p(t), \quad \text{(50a)}$$

$$z(t) = \begin{bmatrix} C \\ HK \end{bmatrix} \xi(t), \quad \text{(50b)}$$

corresponding to the system in equation (50) *with feedback control* $u(t) = K\xi(t)$, *satisfies following condition.*

$$\sup_{\substack{v \in L_2^{\bar{l}}(0,\ \infty),\\ v \neq 0,\ \xi(0)=0}} \frac{\|z\|_2^2}{\|v\|_2^2} := \sup_{\substack{v \in L_2^{\bar{l}}(0,\ \infty),\\ v \neq 0,\ x(0)=0}} \frac{E\int_0^{+\infty}[\xi^T(t)C^TC\xi(t)+u^T(t)u(t)]dt}{E\int_0^{+\infty} v^T(t)v(t)dt} < \gamma^2. \quad (51)$$

The following result is well known [45, 46].

Lemma 6. *Suppose that Assumption 4 is satisfied. The stochastic H_∞ state-feedback control problem has a solution if and only if there exists a symmetric non-negative definite solution Z_ε to the following SMARE*

$$\boldsymbol{G}(Z_\varepsilon) := A_\varepsilon^T Z_\varepsilon + Z_\varepsilon A_\varepsilon + \sum_{p=1}^{M} A_{p\varepsilon}^T Z_\varepsilon A_{p\varepsilon}$$

$$-Z_\varepsilon(B_\varepsilon B_\varepsilon^T - \gamma^{-2} D_\varepsilon D_\varepsilon^T)Z_\varepsilon + C^TC = 0 \quad (52)$$

such that the stochastic system

$$d\xi(t) = [A_\varepsilon - B_\varepsilon B_\varepsilon^T Z_\varepsilon + \gamma^{-2} D_\varepsilon D_\varepsilon^T Z_\varepsilon]\xi(t)dt + \sum_{p=1}^{M} A_{p\varepsilon}\xi(t)dw_p(t) \quad (53)$$

is EMSS.

The controller solving this H_∞ problem is given by equation (54).

$$u(t) = K\xi(t) = -B_\varepsilon^T Z_\varepsilon \xi(t). \quad (54)$$

6.1 Asymptotic Structure of SMARE

In this section, we need to first analyze the asymptotic structure of SMARE (52) to obtain the controller. In order to simplify the presentation, the following matrices are defined.

$$\hat{S}_{\varepsilon} := B_{\varepsilon}B_{\varepsilon}^T - \gamma^{-2}D_{\varepsilon}D_{\varepsilon}^T = \begin{bmatrix} \hat{S}_{00} & \hat{S}_{0f}\Pi_{\varepsilon}^{-1} \\ \Pi_{\varepsilon}^{-1}\hat{S}_{0f}^T & \Pi_{\varepsilon}^{-1}\hat{S}_f\Pi_{\varepsilon}^{-1} \end{bmatrix},$$

$$\hat{S}_{0f} := \begin{bmatrix} \hat{S}_{01} & \cdots & \hat{S}_{0N} \end{bmatrix}, \ \hat{S}_f := \textbf{block diag}\begin{pmatrix} \hat{S}_{11} & \cdots & \hat{S}_{NN} \end{pmatrix}.$$

Let $\bar{Z}_{00}$, $\bar{Z}_{f0}$ and $\bar{Z}_f$ be the limiting solutions of the above SMARE (52) as $\mu \to +0$, $\varepsilon_i \to +0$, $i = 1, \ \ldots, \ N$, then we obtain the following reduced-order equations (55).

$$\bar{Z}_{00}A_0 + A_0^T\bar{Z}_{00} + \bar{Z}_{f0}^TA_{f0} + A_{f0}^T\bar{Z}_{f0} + \sum_{p=1}^{M}A_{p00}^T\bar{Z}_{00}A_{p00}$$
$$-\bar{Z}_{00}S_{00}\bar{Z}_{00} - \bar{Z}_{f0}^TS_f\bar{Z}_{f0} - \bar{Z}_{00}S_{0f}\bar{Z}_{f0} - \bar{Z}_{f0}^TS_{0f}^T\bar{Z}_{00} + Q_{00} = 0, \quad \text{(55a)}$$
$$A_{f0}^T\bar{Z}_f + \bar{Z}_{00}A_{0f} + \bar{Z}_{f0}^TA_f - \bar{Z}_{00}S_{0f}\bar{Z}_f - \bar{Z}_{f0}^TS_f\bar{Z}_f + Q_{0f} = 0, \quad \text{(55b)}$$
$$\bar{Z}_f^TA_f + A_f^T\bar{Z}_f - \bar{Z}_f^TS_f\bar{Z}_f + Q_f = 0, \quad \text{(55c)}$$

First, the following AREs are introduced.

$$\bar{Z}_{ii}^*A_{ii} + A_{ii}^T\bar{Z}_{ii}^* - \bar{Z}_{ii}^*\hat{S}_{ii}\bar{Z}_{ii}^* + Q_{ii} = 0, \ i = 1, \ \ldots, N. \quad (56)$$

Moreover, let us define the following sets.

$\Gamma_{f_i} = \{\gamma > 0|$ the ARE (56) with $\hat{S}_{ii} = B_{ii}B_{ii}^T - \gamma^{-2}D_{ii}D_{ii}^T$ has a positive semidefinite and stabilizing solution $\bar{Z}_{ii}^*\}$, $i = 1, \ \ldots, N$.

Assumption 5. *The sets Γ_{f_i} are not empty.*

Lemma 7. *Under Assumption 5, the asymmetric ARE* (55c) *admits a unique symmetric positive semidefinite stabilizing solution $\bar{Z}_f$ which can be written as*

$$\bar{Z}_f^* := \textbf{block diag}\begin{pmatrix} \bar{Z}_{11}^* & \cdots & \bar{Z}_{NN}^* \end{pmatrix}. \quad (57)$$

Assumption 5 ensures that $A_{ii} - \hat{S}_{ii}\bar{Z}_{ii}^*$, $i = 1, \ \ldots, \ N$ are nonsingular. Substituting the solution of (55c) into (55b) and substituting $\bar{Z}_{f0}^*$ into (55a) and making some lengthy calculations, we obtain the following zeroth-order equations (58).

$$\bar{Z}_{00}^{*}\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}^{T}\bar{Z}_{00}^{*} + \sum_{p=1}^{M} A_{p00}^{T}\bar{Z}_{00}^{*}A_{p00} - \bar{Z}_{00}^{*}\hat{\boldsymbol{S}}\bar{Z}_{00}^{*} + \hat{\boldsymbol{Q}} = 0, \tag{58a}$$

$$\bar{Z}_{i0}^{*T} := \begin{bmatrix} \bar{Z}_{ii}^{*} & -I_{n_i} \end{bmatrix} \hat{T}_{ii}^{-1}\hat{T}_{i0} \begin{bmatrix} I_{n_0} \\ \bar{Z}_{00}^{*} \end{bmatrix}, \tag{58b}$$

$$\bar{Z}_{ii}^{*}A_{ii} + A_{ii}^{T}\bar{Z}_{ii}^{*} - \bar{Z}_{ii}^{*}\hat{S}_{ii}\bar{Z}_{ii}^{*} + Q_{ii} = 0, \tag{58c}$$

where $\bar{Z}_{f0}^{*} := \begin{bmatrix} \bar{Z}_{10}^{*T} & \cdots & \bar{Z}_{N0}^{*T} \end{bmatrix}^{T}$,

$$\begin{bmatrix} \hat{\boldsymbol{A}} & -\hat{\boldsymbol{S}} \\ -\hat{\boldsymbol{Q}} & -\hat{\boldsymbol{A}}^{T} \end{bmatrix} := \hat{T}_{00} - \sum_{j=1}^{N} \hat{T}_{0j}\hat{T}_{jj}^{-1}\hat{T}_{j0},$$

$$\hat{T}_{00} := \begin{bmatrix} A_0 & -\hat{S}_{00} \\ -Q_{00} & -A_0^{T} \end{bmatrix},\ \hat{T}_{0i} := \begin{bmatrix} A_{0i} & -\hat{S}_{0i} \\ -Q_{0i} & -A_{i0}^{T} \end{bmatrix},$$

$$\hat{T}_{i0} := \begin{bmatrix} A_{i0} & -\hat{S}_{0i}^{T} \\ -Q_{0i}^{T} & -A_{0i}^{T} \end{bmatrix},\ \hat{T}_{ii} := \begin{bmatrix} A_{ii} & -\hat{S}_{ii} \\ -Q_{ii} & -A_{i}^{T} \end{bmatrix},\ i = 1,\ \ldots, N.$$

Remark 4. *For each $i \in \{1, \ldots, N\}$ equation* (56) *is a Riccati equation arising in connection with the deterministic H_∞ problem. Hence, if Γ_{f_i} is not empty then $\Gamma_{f_i} = (\gamma_{f_i}, \infty)$. On the other hand, if $\gamma \in \Gamma_{f_i}$ then the matrix $A_{ii} - \hat{S}_{ii}\bar{Z}_{ii}^{*}$ is a stable matrix. Therefore the hamiltonian $\hat{T}_{ii}$ is invertible.*

The ARE (58c) produces a positive semidefinite solution if γ is sufficiently large. Hence, let us define the set.
$\Gamma_s = \{\gamma > 0|$ the SARE (58a) has a positive semidefinite and stabilizing solution $\bar{Z}_{00}^{*}\}$.

We introduce the assumption:

Assumption 6. *The set Γ_s is not empty and it has the form $\Gamma_s = (\gamma_s, \infty)$.*

Remark 5. a) *In the considered general case it is not clear how the coefficients $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{S}}$, $\hat{\boldsymbol{Q}}$ are depending upon γ. That is why we have to introduce as an assumption the fact that the set Γ_s takes the form of a right unbounded interval. It is worth mentioning that this happens if all matrices A_{ii} are invertible.*

b) *The fact that* $\bar{Z}_{00}^*$ *is the stabilizing solution of* (58a) *means that the trajectory* $x(t) = 0$ *of the Itô differential equation*

$$dx(t) = [\hat{\boldsymbol{A}} - \hat{\boldsymbol{S}}\bar{Z}_{00}^*]x(t)dt + \sum_{p=1}^{M} A_{p00}x(t)dw_p(t) \tag{59}$$

is EMSS. This is equivalent to the fact that the Lyapunov operator $X \to [\hat{\boldsymbol{A}} - \hat{\boldsymbol{S}}\bar{Z}_{00}^*]^T X + X[\hat{\boldsymbol{A}} - \hat{\boldsymbol{S}}\bar{Z}_{00}^*] + \sum_{p=1}^{M} A_{p00}^T X A_{p00}$ *are located in the half plane* $\mathbf{Re}\lambda < 0$. *This means that* (59) *is true.*

The limiting behavior of Z_ε is described by the following theorem.

Theorem 9. *Under Assumptions 5 and 6, if a parameter* $\gamma > \bar{\gamma} := \max\{\gamma_s,\ \gamma_{f_1},\ \ldots\ ,\gamma_{f_N}\}$ *is selected, there exists a small* σ^* *such that for all* $\|\nu\| \in (0,\ \sigma^*)$, *the SMARE* (52) *admits the unique symmetric positive semidefinite stabilizing solution* Z_ε *for stochastic system* (47) *which can be written as*

$$\begin{aligned} Z_\varepsilon &= \Phi_\varepsilon \begin{bmatrix} \bar{Z}_{00}^* + O(\|\nu\|) & [\bar{Z}_{f0}^* + O(\|\nu\|)]^T \Pi_\varepsilon \\ \bar{Z}_{f0}^* + O(\|\nu\|) & \bar{Z}_f^* + O(\|\nu\|) \end{bmatrix} \\ &= \begin{bmatrix} \bar{Z}_{00}^* + O(\|\nu\|) & [\bar{Z}_{f0}^* + O(\|\nu\|)]^T \Pi_\varepsilon \\ \Pi_\varepsilon[\bar{Z}_{f0}^* + O(\|\nu\|)] & \Pi_\varepsilon[\bar{Z}_f^* + O(\|\nu\|)] \end{bmatrix}, \end{aligned} \tag{60}$$

where $\nu := \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_N & \mu \end{bmatrix} \in \Re^{N+1}$.

It should be noted that there is no solution of to the SMARE (52) as long as there are no positive semi-definite solutions $\bar{Z}_{ii}$ to the SARE (58c). Conversely, the asymptotic structure of the solution to the SMARE (52) can be established by using the reduced-order solution $\bar{Z}_{ii}$ of the SARE (58c) via an implicit function theorem. Therefore, the existence of the reduced-order solution $\bar{Z}_{ii}$ of the SARE (58c) will play an important role in this study. In this case, it is easy to verify that the magnitude of the disturbance attenuation level γ_{f_i} influences the existence of the reduced-order solution $\bar{Z}_{ii}$. In fact, when γ_{f_i} tends to zero, it is hard to obtain the reduced-order solution $\bar{Z}_{ii}$ except for the special case. Finally, the problem considered in this study is restricted for the disturbance attenuation level γ_{f_i} such that the reduced-order SAREs (58c) have the solutions $\bar{Z}_{ii}$.

6.2 Newton's Method

Let us consider Newton's method (61).

$$Z_\varepsilon^{(n+1)}(A_\varepsilon - \hat{S}_\varepsilon Z_\varepsilon^{(n)}) + (A_\varepsilon - \hat{S}_\varepsilon Z_\varepsilon^{(n)})^T Z_\varepsilon^{(n+1)} \\ + \sum_{p=1}^{M} A_{p\varepsilon}^T Z_\varepsilon^{(n+1)} A_{p\varepsilon} + Z_\varepsilon^{(n)} \hat{S}_\varepsilon Z_\varepsilon^{(n)} + Q = 0, \tag{61}$$

where $n = 0,\ 1,\ \dots$, and the initial conditions are chosen as follows.

$$Z_\varepsilon^{(0)} := \Phi_\varepsilon \left[\begin{array}{cc} \bar{Z}_{00}^* & \bar{Z}_{f0}^{*T}\Pi_\varepsilon \\ \bar{Z}_{f0}^* & \bar{Z}_f^* \end{array} \right] = \Phi_\varepsilon \bar{Z}. \tag{62}$$

Using the asymptotic structure of (60), it should be noted that the initial condition is chosen as (62).

The algorithm represented by equation (61) has the feature given in the following theorem for the MSPSS.

Theorem 10. *Suppose that Assumptions 5 and 6 are satisfied. If the parameter-independent reduced-order SARE* (58c) *has a positive semidefinite solution, there exists a positive scalar $\hat{\sigma}$ such that for all $\varepsilon \in H$ with $0 < \|\varepsilon\| \le \hat{\sigma}$, the iterative algorithm represented by equation* (61) *converges to the exact solution of Z_ε with a rate equal to that of quadratic convergence; here, $Z_\varepsilon^{(n)}$ is positive semidefinite. Moreover, the convergence solutions equal those of Z_ε in the SMARE* (52) *in the neighborhood of the initial condition $Z_\varepsilon^{(0)} = \Phi_\varepsilon \bar{Z}$. In other words, the following condition is satisfied.*

$$\|Z_\varepsilon^{(n)} - Z_\varepsilon\| = \frac{(2\hat{\theta})^{2^n}}{2^n \hat{\beta} \hat{\boldsymbol{L}}} = O(\|\nu\|^{2^n}),\ n = 0,\ 1,\ \dots , \tag{63}$$

where
$\hat{\boldsymbol{L}} = 2\|\hat{S}_\varepsilon\| < \infty$, $\hat{\beta} = \|[\nabla \boldsymbol{G}(Z_\varepsilon^{(0)})]^{-1}\|$, $\hat{\theta} = \hat{\beta}\hat{\eta}\hat{\boldsymbol{L}} < 2^{-1}$ $\hat{\eta} = \|[\nabla \boldsymbol{G}(Z_\varepsilon^{(0)})]^{-1}\| \cdot \|\boldsymbol{G}(Z_\varepsilon^{(0)})\|$.

7 Simulation Example

In order to demonstrate the efficiency of the stochastic H_∞ control for MSPSS, we present results for practical multiarea electric energy systems.

The state variable model of the megawatt-frequency control problem was developed in [47].

In developing the state space model, the following basis equations were used:

$$
\begin{aligned}
\Delta P_{\mathrm{tie}i} &= \sum_{v} T_{iv}^{*}\left(\int \Delta f_i dt - \int \Delta f_v dt\right), \\
\Delta P_{gi} - \Delta P_{di} &= \frac{2H_i}{f^*}\frac{d}{dt}\Delta f_i + D_i \Delta f_i + \Delta P_{\mathrm{tie}i}, \\
\frac{d}{dt}\Delta P_{gi} &= -\frac{1}{T_{ti}}\Delta P_{gi} + \frac{1}{T_{ti}}\Delta X_{gvi}, \\
\frac{d}{dt}\Delta X_{gvi} &= -\frac{1}{T_{gvi}}\Delta X_{gvi} - \frac{1}{T_{gvi}R_i}\Delta f_i + \frac{1}{T_{gvi}}\Delta P_{ci}.
\end{aligned}
$$

Some system parameters used in our study are referred to [47] for details.

For a two-area MSPSS, the following state, control and disturbance variables can be defined.

$$
\begin{aligned}
\xi(t) &:= \left[\, \int \Delta P_{\mathrm{tie}1} dt \; \int \Delta f_1 dt \;\; \Delta f_1 \; \int \Delta f_2 dt \;\; \Delta f_2 \; \Delta P_{g1} \;\; \Delta P_{g2} \middle| \Delta X_{gv1} \middle| \Delta X_{gv2} \,\right]^T \\
&= \left[\; x(t) \mid z_1(t) \mid z_2(t) \;\right]^T, \\
u(t) &:= \left[\; \Delta P_{c1} \quad \Delta P_{c2} \;\right]^T, \; v(t) := \left[\; \Delta P_{d1} \quad \Delta P_{d2} \;\right]^T.
\end{aligned}
$$

The following system data were used for the numerical calculation.

$$
\begin{aligned}
&P_{r1} = P_{r2} = 2000 \text{ [MW]}, \; H_1 = H_2 = 5 \text{ [sec]}, \\
&D_1 = D_2 = 8.33 \times 10^{-3} \text{ [puMW/Hz]}, \\
&T_{t1} = T_{t2} = 0.3 \text{ [sec]}, \; T_{gv1} = 0.030, \\
&T_{gv2} = 0.029 \text{ [sec]}, \; \delta_1^* - \delta_2^* = 60 \text{ [degree]}, \\
&R_1 = R_2 = 2.4 \text{ [Hz/puMW]}, \; f^* = 60 \text{ [Hz]}, \\
&T_{12}^* = 0.315 \text{ [puMW]}, \; \Delta P_{di} = 0.1 \text{ [puMW]}.
\end{aligned}
$$

$$
A_{00} = \begin{bmatrix}
0 & 0.315 & 0 & -0.315 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1.888 & -0.0498 & 1.888 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1.888 & 0 & -1.888 & -0.0498 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3.333 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3.333
\end{bmatrix},
$$

$$A_{01} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3.333 \\ 0 \end{bmatrix},\ A_{02} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 0 \\ 3.333 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0.41666 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.41666 & 0 & 0 \end{bmatrix},$$

$$A_{20} = \begin{bmatrix} 0 & 0 & 0.41666 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.41666 & 0 & 0 \end{bmatrix},\ A_{11} = A_{22} = -1,$$

$$A_{100} = \textbf{block diag}\begin{pmatrix} 0 & 0 & 0.00249 & 0 & 0.00249 & 0 & 0 \end{pmatrix},$$

$$A_{110} = A_{120} = 0 \in \Re^{1\times 7},\ A_{111} = A_{112} = A_{122} = 0,\ B_{01} = B_{02} = 0 \in \Re^{7\times 1},$$

$$B_{11} = 1,\ B_{22} = 1,\ D_{01} = \begin{bmatrix} 0 & 0 & -0.6 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$D_{02} = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.6 & 0 & 0 \end{bmatrix}^T,\ D_{11} = D_{22} = 0,$$

$$Q = \textbf{block diag}\,(I_7\ 0.25I_2)\,.$$

The system matrices are given by the top of this page. It is assumed that time constant of the governors represents the small singular perturbations. Hence, small parameters are $T_{gv1} := \varepsilon_1 = 0.030$ and $T_{gv2} := \varepsilon_2 = 0.029$. Moreover, it should be noted that $\mu = 0$.

It should be noted that the deterministic disturbance distribution $v(t) := [\Delta P_{d1}\ \Delta P_{d2}]^T = [0.1\ 0.1]^T$ and the state-dependent noise related to the load frequency constant [47] are both considered compared with the existing results [48, 49]. We suppose that the error in the load frequency constant is within 5% of the nominal value. Therefore, the proposed design method is very useful because the resulting strategy can be implemented on more practical MSPSS.

For every boundary value $\gamma > \bar{\gamma} := \max\{\gamma_s,\ \gamma_{f_1},\ \gamma_{f_2}\} = 2.2608e - 1$, the SMARE (52) has a positive definite stabilizing solution because the AREs (55c) and the SARE (55a) have a positive definite solution, where $\gamma_s = 2.2608e - 1$, $\gamma_{f_1} = \gamma_{f_2} = \infty$.

Now, we choose $\gamma = 0.3\ (> \bar{\gamma})$ to solve the MSARE (7). The efficiency of Newton's method (61) is demonstrated. It is easy to verify that algorithm (61) converges to the exact solution with an accuracy of $\|\boldsymbol{G}(Z_{\varepsilon}^{(n)})\| < 1.0e{-}11$

after five iterations.

Table 1. Errors per iterations.

n	$\|\boldsymbol{G}(Z_\varepsilon^{(n)})\|$
0	1.5667
1	$4.2489e-01$
2	$3.3631e-03$
3	$2.0470e-05$
4	$1.5710e-11$
5	$9.1508e-12$

In order to verify the accuracy of the solution, the remainder per iteration is substituted as $Z_\varepsilon^{(n)}$ into SMARE (52). In Table 1, the results of the error $\|\boldsymbol{G}(Z_\varepsilon^{(n)})\|$ per iteration are given. It can be seen that algorithm (61) yields quadratic convergence. Using the obtained iterative solution, the high-order approximate stochastic H_∞ controller is given as follows.

$$u^{(5)}(t) = \begin{bmatrix} 1.5893 & 9.4531e-1 & 4.1393 & 1.6120 & 1.8547e-1 \\ -7.8321e-1 & 1.7522e-3 & 2.3204e-1 & 1.1581 & 9.5872e-1 \end{bmatrix.$$
$$\left. \begin{matrix} 4.2214 & -2.8374e-2 & 4.6816e-1 & 2.1536e-2 \\ 2.6205e-1 & 9.3331e-2 & 2.2279e-2 & 2.6668e-1 \end{matrix} \right] \xi(t).$$

In addition, when the small parameters $\varepsilon_i,\ i = 1,\ 2$ are unknown, we can obtain the parameter-independent control as follows by using the similar technique in section 3.3.

$$u_{\text{app}}(t) = \left[\begin{matrix} 1.3707 & 8.7785e-1 & 3.5978 & 1.3178 & 1.3358e-1 \\ -7.8269e-1 & -4.5742e-2 & 1.8744e-1 & 1.1557 & 9.1813e-1 \end{matrix} \right.$$
$$\left. \begin{matrix} 3.5938 & -2.5123e-2 & 1.1803e-1 & 0 \\ 2.1534e-1 & 1.0543e-1 & 0 & 1.1803e-1 \end{matrix} \right] \xi(t).$$

This control will also be reliable because they seem to be close.

8 Conclusion

The existing results and recent research trends in the various multimodeling analysis and design methods have been briefly summarized. A thorough study of both the parameter-independent methodology and the numerical algorithms revealed the properties of the different methods have been given.

The following conclusion can be drawn: When the small perturbation parameters ε_i are not known, it is strongly recommended that the two-time-scale decomposition method or descriptor systems approach be used. On the other hand, as long as the small perturbation parameters ε_i are known, effort should be made towards finding the exact solutions by means of numerical algorithm. In particular, since the closed-loop solution of the reduced Nash problem depends on the path, the required solution has to be solved numerically.

This survey has mostly concentrated on some classical and recent developments in parameter-independent and computational methods for designing the strategy. Although the choice of topics was necessarily somewhat limited, there are some topics which deserve further attention. For example, the mathematical model described by Itô, i.e. differential equations with Markovian switching in the multimodel situation, is very interesting. This problem will be addressed in future investigations.

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